

Lyapunov spectra for all symmetry classes of quasi-one-dimensional disordered systems

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Abstract

A random phase property is proposed for products of random matrices drawn from one of the classical groups associated to the 10 Cartan symmetry classes. It allows to calculate the Lyapunov spectrum explicitly in a perturbative regime. These results apply to quasi-one-dimensional random Dirac operators which can be constructed for each of the symmetry classes. For classes corresponding to quantum Hall systems, quantum spin Hall systems and \mathbb{Z}_2 -topological superconductors the random Dirac operators have vanishing Lyapunov exponents and almost surely an absolutely continuous spectrum.

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The theory of products of random matrices has successfully been applied to the analysis of localization effects in quasi-one-dimensional disordered systems. In the mathematical literature [BL], particular focus has traditionally been put on time-reversal invariant systems without spin, for which the transfer matrices are in the symplectic group. On the other hand, it is well known from work of Altland and Zirnbauer [AZ] (see also [SRFL, RSFL]) that there are also disordered systems of non-interacting Fermions with time-reversal, sublattice (also called chiral) and/or particle-hole symmetries (denoted by TRS, SLS and PHS below). In dimension 1 this leads to transfer matrices in other classical groups (see Section 1 below for terminology), and actually each of the non-exceptional groups of the Cartan classification appears as the transfer matrix group of an adequate quasi-one-dimensional model. For example, an SLS appears for lattice operators typically at special energies when there is a sublattice structure. The best known example is probably the Dyson model with off-diagonal disorder, for which the transfer matrices at zero energy have such a chiral symmetry (it is in the chiral orthogonal class). On the other hand, the Bogoliubov-deGennes (BdG) classes describe quasi-particles in dirty superconductors. These quasi-particles can be either particles or holes, and the Hamiltonians then have a supplementary PHS. Random Dirac operators with adequate physical symmetries allow to construct concrete models having transfer matrices (fundamental solutions) in each class [BMSA, BFGM]. The interplay between physical symmetries of these Hamiltonians and

the symmetries of the transfer matrix group are reviewed in Sections 3 and 4 below. Of course, apart from this physically motivated application the study of random products in each of the classical groups is of intrinsic mathematical interest as well.

The most interesting quantities associated to such random products are the Lyapunov exponents, which are the exponential growth rates of their singular values. The collection of these exponents constitute the Lyapunov spectrum. In the physics literature, the approach most widely used for their study is based on the Dorokhov-Mello-Pereyra-Kumar (DMPK) equations [Dor, MPK]. These equations are universal (in the sense that they do not contain model dependent information other than the above mentioned symmetry properties), but their derivation relies on the so-called maximal entropy Ansatz. A recently proposed alternative approach [RS2] is closer to the theory of products of random matrices and the numerical transfer matrix method. One considers the random action of the transfer matrices on the compact Grassmannian. This leads to a Markov process and then one supposes that the *random phase property* (RPP) holds, namely that the invariant measure of the Markov process is given by geometric invariant measure on the Grassmannian. In [RS2] the RPP was only studied for the three standard symmetry classes, and here this notion is extended to treat the other symmetry classes of BdG and chiral type. Models for which the RPP holds exactly can easily be constructed, see Section 2. As discussed in [RS2], the RPP is considerably weaker than the maximal entropy Ansatz and it is susceptible to numerical verification in concrete models.

Whenever the RPP holds, the calculation of the Lyapunov spectrum reduces to the evaluation of the averages of certain functions over the invariant measures on the Grassmannian. These invariant measures are given in terms of the Haar measure on the maximal compact subgroups of the given classical group, and all these compact subgroups are in turn given in terms of the unitary, orthogonal or symplectic group. While in principle it is known how to calculate these averages [Wei, CS, CSt], it becomes feasible only in a perturbative regime of small randomness. Then one only needs the second and some fourth moments of these Haar measures. The necessary formulas are collected in the Appendix. The outcome of the rather tedious calculations are nice and compact formulas for the Lyapunov spectrum in a perturbative regime. They are listed for each symmetry class separately in the subsections of Section 2. The Lyapunov exponents $\gamma_p \geq \gamma_{p+1}$ in a system with N channels and a centered random perturbation of strength are in all cases roughly given by the same formula

$$\gamma_p \sim \lambda^2 \frac{N - \eta p}{N^2} + \mathcal{O}(\lambda^3) ,$$

with $\eta = 1$ or $\eta = 2$, but particular focus is on calculating the prefactors in this formula in each symmetry class. This also allows to calculate the smallest non-negative Lyapunov exponent which in physical models is interpreted as the inverse localization length. The precise form of the prefactors is particularly important for the study of systems with a small number of channels. These general results on the Lyapunov spectrum apply directly to the model Dirac operators of Section 4.

It is possible that the products of random matrices have vanishing Lyapunov exponents which are symmetry enforced, namely they result from the defining relations of the classical

group considered. The most basic example of this type are products in the group $U(N, M)$ with $M > N$ which have at least $M - N$ vanishing Lyapunov exponents. This corresponds to a quantum Hall system with $N - M$ edge channels (the quasi-one-dimensional operator then gives an effective description of the edge states). Subclasses correspond to the thermal and spin quantum Hall effect. Other interesting cases concern quasi-one-dimensional Hamiltonians in the Cartan class AII and DIII which describe boundary states in two-dimensional quantum spin Hall systems and time reversal invariant dirty superconductors respectively. The parity of their channel number N can be understood as a \mathbb{Z}_2 -invariant and, if it is non-trivial (namely, N is odd), then there are again at least two vanishing Lyapunov exponents. This leads to delocalization and to absolutely continuous spectral measures. The case AII was studied in detail in [SS2]. Here in Section 5 we discuss how these results transpose to the case DIII as well as quantum Hall systems.

1 A list of the classical groups

The classical groups in the sense of Weyl incorporate certain families of matrix groups. The list is given in the first chapter of [GW]. There is also a close connection to the Cartan classification of symmetric spaces because each classical group from the list below corresponds to a non-compact Lie algebra in duality with a classical compact symmetric space [Hel]. As such there is a Cartan label associated to each classical group and this label has been used heavily in the physics community [AZ, BFGM]. For this reason, we also choose to attach the Cartan label to each classical group \mathcal{G} , but at the same time also the classical terminology is recalled in each case below. Because it will be heavily used later one, we also recall the maximal compact subgroup \mathcal{U} for each classical group \mathcal{G} as well as peculiarities of the spectral theory in \mathcal{G} . The groups \mathcal{G} are listed in Table 1, albeit in a different order than we go through them here. The groups will be described using the matrices

$$J = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad I = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad (1)$$

where the identity $\mathbf{1}$ is of adequate size, mostly an $N \times N$ matrix. These are real matrices satisfying $J^* = J$, $K^* = K$, $I^* = -I$ and $\mathbf{1} = J^2 = -I^2 = K^2$ which is why we call J and K an even, but I an odd symmetry. They do not commute, and one has $KJ = I$. Of course, one could use Pauli matrices instead. Writing $\imath = \sqrt{-1}$, the three involutions J , K and $\imath I$ are isospectral, and one can pass from one to another by a unitary transformation. In particular, the Cayley transformation C achieves the following

$$C^* J C = \imath I, \quad C J C^* = K, \quad C = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & -\imath \mathbf{1} \\ \mathbf{1} & \imath \mathbf{1} \end{pmatrix}. \quad (2)$$

Moreover, $C^* I C = \imath K$, $C^t K C = \mathbf{1}$, $C K C^t = \imath J$ and $C I C^t = -\imath I$. Among the possible realizations of the classical groups we choose ones for which the maximal compact subgroup takes a particularly simple form, since this will allow to compute averages over these subgroups later on.

1.1 Class A

In Class A there are no further symmetries imposed. Thus $\mathcal{G}^A = \text{GL}(N, \mathbb{C})$ is the set of all invertible $N \times N$ matrices. The maximal compact subgroup $\mathcal{U}^A = \text{U}(N)$ is given by the unitary matrices.

1.2 Class AI

Class AI contains the subgroup of \mathcal{G}^A composed of the real matrices (invariant under the operation of complex conjugation):

$$\mathcal{G}^{\text{AI}} = \{T \in \text{GL}(N, \mathbb{C}) \mid \bar{T} = T\} = \text{GL}(N, \mathbb{R}).$$

Clearly the maximal compact subgroup $\mathcal{U}^{\text{AI}} = \text{O}(N)$ is composed of the orthogonal matrices.

1.3 Class AII

The definition of Class AII is

$$\mathcal{G}^{\text{AII}} = \{T \in \text{GL}(2N, \mathbb{C}) \mid I^* \bar{T} I = T\} = \text{U}^*(2N).$$

Writing out the symmetry one readily checks that $T \in \text{U}^*(2N)$ is of the form $T = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$ with entries given by $N \times N$ complex matrices. One obtains an $N \times N$ quaternion matrix from T by setting $\mathcal{F}(T) = A + Bj$ where j is the second of the standard unit quaternions i, j, k satisfying $i^2 = j^2 = k^2 = ijk = -1$. This map \mathcal{F} is a $*$ -isomorphism from $\text{U}^*(2N)$ to the group $\text{GL}(N, \mathbb{H})$ of invertible quaternion matrices. Thus the classes A, AI, AII simply correspond to the invertible matrices over the field $\mathbb{C}, \mathbb{R}, \mathbb{H}$ respectively. The maximal compact subgroup of $\text{U}^*(2N)$ is

$$\mathcal{U}^{\text{AII}} = \text{U}^*(2N) \cap \text{U}(2N) = \left\{ \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \mid A^* A + B^t \bar{B} = \mathbf{1}, \quad A^* B^t = B^t \bar{A} \right\} = \text{SP}(2N).$$

The spectral theory of $T \in \text{U}^*(2N)$ has the following feature. If $Tv = \lambda v$ for some $v \in \mathbb{C}^{2N}$ and $\lambda \in \mathbb{C}$, then $T I \bar{v} = \bar{\lambda} I \bar{v}$ and furthermore the vectors v and $I \bar{v}$ are linearly independent. Indeed, suppose $v = \begin{pmatrix} a \\ b \end{pmatrix}$ satisfies $v = \mu I \bar{v}$ for some non-vanishing $\mu \in \mathbb{C}$. Then $a = -\mu \bar{b}$ and $b = \mu \bar{a}$, which combined imply $a = -|\mu|^2 a$ so that $a = b = 0$. This implies that the real spectrum of $T \in \text{U}^*(2N)$ has even geometric multiplicity, a fact that is called Kramers' degeneracy in the physics literature. Actually, all the above also holds for all (possibly non-invertible) matrices in the Lie algebra $\mathfrak{u}^*(2N)$ (which by the above map \mathcal{F} is mapped to all $N \times N$ quaternion matrices).

1.4 Class AIII

The group of Class AIII is defined by:

$$\mathcal{G}^{\text{AIII}} = \{T \in \text{GL}(2N, \mathbb{C}) \mid T^* J T = J\} = \text{U}(N, N),$$

for which maximally compact subgroup is

$$\mathcal{U}^{\text{AIII}} = \text{U}(N, N) \cap \text{U}(2N) = \left\{ \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \middle| V, W \in \text{U}(N) \right\} = \text{U}(N) \times \text{U}(N) .$$

Elements of $\text{U}(N, N)$ are often also called J -unitaries. Their spectrum has a reflection property around the unit circle \mathbb{S}^1 because $T - \lambda \mathbf{1} = \lambda J(T^*)^{-1}(T - (\bar{\lambda})^{-1} \mathbf{1})^* J$ so that the invertibility of $T - \lambda \mathbf{1}$ is indeed equivalent to the invertibility of $T - (\bar{\lambda})^{-1} \mathbf{1}$. Note, however, that in general the two eigenvectors satisfy no relation so that one also cannot conclude that each eigenvalue on the unit circle is degenerate. Another important feature of the spectral theory in $\text{U}(N, N)$ is that the eigenvectors v and w to eigenvalues λ and μ with $\bar{\mu}\lambda \neq 1$ are J -orthogonal, namely $w^* J v = 0$. Indeed, $Tv = \lambda v$ and $Tw = \mu w$ imply $w^* J v = (\mu^{-1} T w)^* J \lambda^{-1} T v = (\bar{\mu} \lambda)^{-1} w^* J v$ which implies the claim. In particular, this shows that eigenvectors v corresponding to eigenvalues off the unit circle are J -isotropic, that is satisfy $v^* J v = 0$.

Class AIII also contains a generalization of the group $\mathcal{G}^{\text{AIII}}$ defined above:

$$\text{U}(N, M) = \{ T \in \text{GL}(N + M, \mathbb{C}) \mid T^* G T = G \} , \quad G = \begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & -\mathbf{1}_M \end{pmatrix} , \quad (3)$$

where, say, $M \geq N$ are possibly different. Then $\text{U}(N, M) \cap \text{U}(N + M) = \text{U}(N) \times \text{U}(M)$ is the maximal compact subgroup. All the above arguments about the spectral theory still apply to $T \in \text{U}(N, M)$. In particular, the spectrum always comes in pairs $\lambda, (\bar{\lambda})^{-1}$ and eigenvectors to eigenvalues λ and μ with $\bar{\mu}\lambda \neq 1$ are J -orthogonal. This implies that there are at most N eigenvalues off the unit circle, or, alternatively stated, at least $M - N$ eigenvalues on the unit circle (here both N and $M - N$ give the algebraic multiplicity). Indeed, let v_1, \dots, v_p be a maximal set of linear independent generalized eigenvectors of eigenvalues of modulus larger than 1. By the above, they then span a p -dimensional G -isotropic subspace, namely a subspace on which G seen as quadratic form vanishes. But the maximal dimension of G -isotropic subspaces is N , so that $p \leq N$.

1.5 Class CI

Class CI consists of the following subgroup of $\mathcal{G}^{\text{AIII}} = \text{U}(N, N)$:

$$\mathcal{G}^{\text{CI}} = \{ T \in \text{GL}(2N, \mathbb{C}) \mid T^* J T = J, \quad K^* \bar{T} K = T \} .$$

Its maximally compact subgroup is

$$\mathcal{U}^{\text{CI}} = \mathcal{G}^{\text{CI}} \cap \text{U}(2N) = \left\{ \begin{pmatrix} V & 0 \\ 0 & \bar{V} \end{pmatrix} \middle| V \in \text{U}(N) \right\} \cong \text{U}(N) .$$

The group \mathcal{G}^{CI} is actually a disguised version of the real symplectic group, namely using the Cayley transform one has

$$C^* \mathcal{G}^{\text{CI}} C = \{ T \in \text{GL}(2N, \mathbb{C}) \mid T^* I T = I, \quad \bar{T} = T \} = \text{SP}(2N, \mathbb{R}) .$$

In this representation the maximal compact subgroup does not take such a simple form. The spectrum of $T \in \mathcal{G}^{\text{CI}}$ has, of course, the \mathbb{S}^1 -reflection property holding for any matrix in \mathcal{G}^{AII} , but, moreover, $Tv = \lambda v$ leads to $T K \bar{v} = \bar{\lambda} K \bar{v}$. Thus the eigenvalues come in quadruples $\lambda, \bar{\lambda}, \lambda^{-1}, (\bar{\lambda})^{-1}$. Let us point out that v and $K \bar{v}$ are, in general, not linearly independent. Therefore one cannot conclude from the above that real eigenvalues are degenerate.

1.6 Class DIII

Also Class DIII is a subgroup of $\mathcal{G}^{\text{AIII}} = \text{U}(N, N)$:

$$\mathcal{G}^{\text{DIII}} = \{ T \in \text{GL}(2N, \mathbb{C}) \mid T^* J T = J, \quad I^* \bar{T} I = T \} .$$

Thus $\mathcal{G}^{\text{DIII}} = \text{U}(N, N) \cap \text{U}^*(2N)$ and one can also see $\mathcal{G}^{\text{DIII}}$ as a subgroup of \mathcal{G}^{AII} . In particular, the map \mathcal{F} described in Section 1.3 is well-defined on $\mathcal{G}^{\text{DIII}}$ and the image $\mathcal{F}(\mathcal{G}^{\text{DIII}})$ is given by those quaternion matrices $\mathcal{F}(T)$ satisfying $\mathcal{F}(T)^* \iota \mathcal{F}(T) = \iota \mathbf{1}$. Its maximally compact subgroup is

$$\mathcal{U}^{\text{DIII}} = \mathcal{G}^{\text{DIII}} \cap \text{U}(2N) = \left\{ \begin{pmatrix} V & 0 \\ 0 & \bar{V} \end{pmatrix} \mid V \in \text{U}(N) \right\} \cong \text{U}(N) .$$

Taking the Cayley transform of $\mathcal{G}^{\text{DIII}}$ one finds, using $C^t I C = \iota I$, one of the classical groups:

$$C^* \mathcal{G}^{\text{DIII}} C = \{ T \in \text{GL}(2N, \mathbb{C}) \mid T^* I T = I, \quad I^* \bar{T} I = T \} = \text{SO}^*(2N) .$$

Here the equation $I^* \bar{T} I = T$ can equivalently be replaced by $T^t T = \mathbf{1}$, because the relation $T^* I T = I$ holds. As to the spectral theory, again the \mathbb{S}^1 -reflection property holds. Furthermore, $Tv = \lambda v$ implies $T I \bar{v} = \bar{\lambda} I \bar{v}$ so that again the spectrum comes in quadruples as in $\mathcal{G}^{\text{CI}} \cong \text{SP}(2N, \mathbb{R})$. Furthermore, because $\mathcal{G}^{\text{DIII}} \subset \mathcal{G}^{\text{AII}}$, the real spectrum has again a Kramers' degeneracy (see Section 1.3).

It is worth mentioning that the two symmetries J and I used in the definition of $\mathcal{G}^{\text{DIII}}$ do not commute. If the matrix size is $4N$ instead of $2N$ which we realize by tensorizing with \mathbb{C}^2 (or equivalently, N is even in the above), then one can also use two commuting symmetries:

$$(CB)^* \mathcal{G}^{\text{DIII}} CB = \{ T \in \text{GL}(4N, \mathbb{C}) \mid T^* I \otimes \mathbf{1} T = I \otimes \mathbf{1}, \quad \mathbf{1} \otimes I^* \bar{T} \mathbf{1} \otimes I = T \} ,$$

where $B = 2^{-\frac{1}{2}} \begin{pmatrix} -\iota & \mathbf{1} \\ \mathbf{1} & \iota \end{pmatrix}$. Indeed, then $B^* = B^{-1} = B$ as well as $B^t \mathbf{1} \otimes I B = -\iota I \otimes \mathbf{1}$ and $B^* I \otimes \mathbf{1} B = -I \otimes \mathbf{1}$. This representation of the group $\mathcal{G}^{\text{DIII}}$ appears as transfer matrices in physical models with odd spin, where then $\mathbf{1} \otimes I$ acts on the spin degree of freedom only (*e.g.* [RS2]).

1.7 Class BDI

Class BDI is yet another subgroup of $\mathcal{G}^{\text{AIII}} = \text{U}(N, N)$:

$$\mathcal{G}^{\text{BDI}} = \{ T \in \text{GL}(2N, \mathbb{C}) \mid T^* J T = J, \quad \bar{T} = T \} = \text{O}(N, N) .$$

Of course, \mathcal{G}^{BDI} is also the subgroup of $\mathcal{G}^{\text{AI}} = \text{GL}(2N, \mathbb{R})$ specified by the relation $T^t J T = J$. Its maximally compact subgroup is

$$\mathcal{U}^{\text{BDI}} = \mathcal{G}^{\text{BDI}} \cap \text{U}(2N) = \left\{ \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \middle| V, W \in \text{O}(N) \right\} = \text{O}(N) \times \text{O}(N) .$$

As matrices in $\text{O}(N, N)$ are real, λ is an eigenvalue if and only if $\bar{\lambda}$ is so. Therefore, the spectrum for matrices in $\text{O}(N, N)$ comes again as complex quadruples, or real pairs λ, λ^{-1} . These pairs are in general not degenerate though (an example for a matrix in $\text{O}(1, 1)$ with simple real eigenvalue pairs is the hyperbolic rotation). Similar as in (3), it is possible to define the groups $\text{O}(N, M)$ with maximal compact subgroup $\text{O}(N) \times \text{O}(M)$. Similar spectral properties as $\text{U}(N, M)$ holds, but this will not be spelled out in detail.

1.8 Class CII

Class CII is also a subgroup of $\mathcal{G}^{\text{AIII}}$, however, with a symmetry that requires tensorizing with another \mathbb{C}^2 so that the matrices are of size $4N \times 4N$. The symmetry in $\text{U}(2N, 2N)$ is denoted by $J \otimes \mathbf{1}$ instead of just J to highlight the difference. Then

$$\mathcal{G}^{\text{CII}} = \{ T \in \text{GL}(4N, \mathbb{C}) \mid T^* J \otimes \mathbf{1} T = J \otimes \mathbf{1}, \quad J \otimes I^* \bar{T} J \otimes I = T \} = \text{SP}(2N, 2N) ,$$

and

$$\mathcal{U}^{\text{CII}} = \mathcal{G}^{\text{CII}} \cap \text{U}(4N) = \left\{ \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \middle| V, W \in \text{SP}(2N) \right\} = \text{SP}(2N) \times \text{SP}(2N) .$$

Obviously, $\mathcal{G}^{\text{CII}} \subset \text{U}(2N, 2N)$, but one also has

$$D^* \mathcal{G}^{\text{CII}} D = \{ T \in \text{GL}(4N, \mathbb{C}) \mid T^* I \otimes I T = I \otimes I, \quad I^* \otimes \mathbf{1} \bar{T} I \otimes \mathbf{1} = T \}$$

where $D = 2^{-\frac{1}{2}} \begin{pmatrix} \mathbf{1} & I \\ I & \mathbf{1} \end{pmatrix}$ and this shows that $D^* \mathcal{G}^{\text{CII}} D \subset \text{U}^*(4N)$. It is not possible, however, to write \mathcal{G}^{CII} as an intersection of $\text{U}(2N, 2N)$ and $\text{U}^*(4N)$ (which was possible in Class DIII).

By the same reasoning as in Class DIII (see Section 1.6) the spectrum of $T \in \text{SP}(2N, 2N)$ comes in quadruples and real eigenvalues always have even geometric multiplicity. However, $Tv = \lambda v$ here implies $T J \otimes I \bar{v} = \bar{\lambda} J \otimes I \bar{v}$. Class DIII has again a generalization $\text{SP}(2N, 2M)$ as in (3), but the obvious details are not given here.

1.9 Class D

Class D is again a subgroup of \mathcal{G}^{A} of matrices of size $N \times N$:

$$\mathcal{G}^{\text{D}} = \{ T \in \text{GL}(N, \mathbb{C}) \mid T^t T = \mathbf{1} \} = \text{O}(N, \mathbb{C}) .$$

Note that this is also a $*$ -group, namely $T^* \in \text{O}(N, \mathbb{C})$ if and only if $T \in \text{O}(N, \mathbb{C})$. Clearly the maximal compact subgroup is $\mathcal{U}^{\text{D}} = \mathcal{G}^{\text{D}} \cap \text{U}(N) = \text{O}(N)$. The relation $T^t T = \mathbf{1}$ implies that the spectrum of T^* is given by the \mathbb{S}^1 -reflection $\lambda \mapsto (\bar{\lambda})^{-1}$ of the spectrum of T . Therefore the spectrum of a self-adjoint element $T = T^* \in \text{O}(N, \mathbb{C})$ is invariant under the map $\lambda \mapsto \lambda^{-1}$.

1.10 Class C

Finally Class C is also a subgroup of \mathcal{G}^A , albeit of size $2N \times 2N$:

$$\mathcal{G}^C = \{T \in \text{GL}(2N, \mathbb{C}) \mid T^t I T = I\} = \text{SP}(2N, \mathbb{C}) .$$

This is also a $*$ -group. The maximal compact subgroup is $\mathcal{U}^C = \mathcal{G}^C \cap \text{U}(2N) = \text{SP}(2N)$. The conclusions for the spectral theory of $T \in \text{SP}(2N, \mathbb{C})$ are the same as in Class D.

2 Lyapunov spectrum with random phase property

Let be given a sequence $(T_n)_{n \geq 0}$ of independently drawn and identically distributed random matrices in one of the above classical groups $\mathcal{G} \subset \text{GL}(N, \mathbb{C})$. For sake of simplicity, these distributions are supposed to be have fine moments (existence of third moment is sufficient for the second order perturbation theory later on). One is then interested in the growth in n of the norm of the products $T_n \cdots T_1$, or what is equivalent, the growth of their largest singular value. As one might expect, it turns out that this growth is exponential with a rate defining the top Lyapunov exponent:

$$\gamma_1 = \lim_{n \rightarrow \infty} \frac{1}{n} \log (\|T_n \cdots T_1\|) .$$

It is known [BL] that the limit converges almost surely to the same value and therefore an expectation can be introduced before the limit on the r.h.s.. Furthermore, instead of calculating an operator norm, an arbitrary vector v , say of unit length, may be inserted so that one actually only needs to calculate the length of vectors:

$$\gamma_1 = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \log (\|T_n \cdots T_1 v\|) .$$

Of course, also the growth rates γ_p of the other singular values are of interest. Together they constitute the Lyapunov spectrum. One convenient way to extract them is to use exterior products $\Lambda^p T_n \cdots T_1$ acting on the p -dimensional Grassmannian identified with the decomposable unit vectors in $\Lambda^p \mathbb{C}^N$ where N is the size of the matrices in N [BL]. Then

$$\sum_{q=1}^p \gamma_q = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \log (\|\Lambda^p T_n \cdots T_1 v_1 \wedge \dots \wedge v_p\|) ,$$

where v_1, \dots, v_p are unit vectors in \mathbb{C}^N which are at least orthogonal (further conditions will be imposed below when \mathcal{G} is not \mathcal{G}^A) and the norm is now calculated in $\Lambda^p \mathbb{C}^N$. It is now very convenient to telescope the r.h.s. so that it becomes a Birkhoff sum. For that purpose let us introduce a sequence $(v^{(p)}(n))_{n \geq 0}$ of decomposable unit vectors in $\Lambda^p \mathbb{C}^N$ by

$$v^{(p)}(n) = \frac{\Lambda^p T_n v^{(p)}(n-1)}{\|\Lambda^p T_n v^{(p)}(n-1)\|} , \quad v^{(p)}(0) = v_1 \wedge \dots \wedge v_p . \quad (4)$$

Indeed, then one has

$$\sum_{q=1}^p \gamma_q = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbf{E} \log (\|\Lambda^p T_k v^{(p)}(k-1)\|) .$$

As the expectation over T_k can immediately be taken in each summand (because $v^{(p)}(k-1)$ is independent of T_k), one is therefore confronted with the calculation of Birkhoff sums of the random dynamical system (4) on the p -dimensional Grassmannian in \mathbb{C}^N which is a compact Riemannian manifold. It is known [BL] that this Markov process has a unique invariant and ergodic measure which we denote by ν . In terms of expectations w.r.t. ν , one then has the so-called Furstenberg formula:

$$\sum_{q=1}^p \gamma_q = \mathbf{E}_T \mathbf{E}_\nu \log (\|\Lambda^p T v^{(p)}\|) . \quad (5)$$

The measure ν is always Hölder continuous and, if the distribution of T is absolutely continuous, then also ν is absolutely continuous [BL]. Here we will be particularly interested in a situation where T_n is of the form

$$T_n = R_n e^{\lambda P_n} , \quad (6)$$

where $R_n \in \mathcal{G}$ is unitary, λ is a real coupling constant and P_n is in the Lie algebra of \mathcal{G} . If the distribution of the R_n and P_n satisfies a certain coupling hypothesis formulated in [SS1] (of Hörmander type), then the invariant measure $\nu = \nu_\lambda$ is in a weak sense and with errors of order λ absolutely continuous w.r.t. to the invariant distribution on $\Lambda^p \mathbb{C}^N$, and even equal to it for the so-called N -orbital model of Wegner [SS1]. This invariant distribution is simply obtained by taking an element $v^{(p)} \in \Lambda^p \mathbb{C}^N$ and randomly rotating it $\Lambda^p U v^{(p)}$ with the unitary U drawn according to the Haar measure on the maximal compact subgroup $\mathcal{U} \subset \mathcal{G}$. If $\mathcal{G} = \mathcal{G}^A = \text{GL}(N, \mathbb{C})$ the measure on $\Lambda^p \mathbb{C}^N$ is independent of the choice of the reference vector $v^{(p)}$. However, if \mathcal{G} is not equal to $\mathcal{G}^A = \text{GL}(N, \mathbb{C})$ so that \mathcal{U} is smaller than $\mathcal{U}^A = \text{U}(N)$, then not the full Grassmannian is attained. Thus then also the choice of the reference vector $v^{(p)}$ becomes relevant. The guiding principle in determining the correct choice is that the vectors v_q can be the eigenvectors of the a positive element in \mathcal{G} (which is $(T_N \cdots T_1)^* T_N \cdots T_1$ in the application above). For example, if $\mathcal{G} = \text{U}(N, N)$, then this imposes that the v_q are J -isotropic and pairwise J -orthogonal. Furthermore, if the spectral theory in \mathcal{G} leads to Kramers' degeneracy, then also the vectors v_q have to respect that structure. The choice of the orthonormal basis will be discussed in detail in each case below. Apart from this (admittedly important) detail, we can now state the central hypothesis of this paper extending that of [RS2]. Strictly speaking it extends [RS2] to other symmetry classes, but is less general because no hyperbolic channels in the terminology of [RS1, RS2] are considered here.

Random phase property (RPP): *The invariant measure ν of the Markov process (4) generated by a random sequence in \mathcal{G} is given by the invariant measure on the Grassmannian obtained by rotating an initial condition $v^{(p)}$ with unitaries drawn from the Haar measure on the maximal compact subgroup $\mathcal{U} \subset \mathcal{G}$.*

We don't expect the RPP to hold exactly except in a model where it is imposed artificially. On the other hand, it may hold approximately in a perturbative situation described above and its implications can also be observed in numerical experiments on particular models, provided one restricts the attention to so-called elliptic channels (*e.g.* [RS1]). As discussed in detail in [RS2], the RPP is strictly weaker than the maximal entropy Ansatz [Dor, MPK] used for the derivation of the DMPK equations. A simple toy model for which the RPP holds exactly (and thus all the below as well) is obtained by requiring R_n in (6) to be Haar distributed on $\mathcal{U} \subset \mathcal{G}$ and then P to be distributed independently according to an arbitrary measure on the Lie algebra of \mathcal{G} . Then it is obvious that the invariant measure ν of the Markov process (4) is given by the Haar measure, namely the RPP holds. In general, provided the RPP holds, the calculation of the Lyapunov spectrum by Furstenberg's formula (5) becomes

$$\sum_{q=1}^p \gamma_q = \mathbf{E}_T \mathbf{E}_{\mathcal{U}} \log (\|\Lambda^p T \Lambda^p U v_1 \wedge \dots \wedge v_p\|) . \quad (7)$$

In a perturbative regime which is of interest for several applications, the evaluation of (7) is now an algebraic exercise. This was actually carried out in detail for the groups $\mathcal{G}^{\text{AIII}}$, \mathcal{G}^{CI} and $\mathcal{G}^{\text{DIII}}$ in [RS2]. Here we also consider the remaining cases which involves calculating averages not only over the unitary, but also the orthogonal and symplectic group. The basic tool will be the following perturbative formula in the case where $T = Re^{\lambda P}$ with R unitary. Let v_1, \dots, v_p be orthonormal vectors in \mathbb{C}^N and set $\pi_p = \sum_{q=1}^p e_q e_q^*$. Then

$$\begin{aligned} & \log (\|\Lambda^p Re^{\lambda P} \Lambda^p U v_1 \wedge \dots \wedge v_p\|) \\ &= \frac{\lambda}{2} \text{Tr} (U^* Q U \pi_p) + \frac{\lambda^2}{4} \text{Tr} (U^* S U \pi_p) - \frac{\lambda^2}{4} \text{Tr} (U^* Q U \pi_p U^* Q U \pi_p) + \mathcal{O}(\lambda^3) , \end{aligned} \quad (8)$$

where

$$Q = P + P^* , \quad S = P^2 + 2P^*P + (P^*)^2 .$$

To prove this, let us begin by recalling the definition of the norm in $\Lambda^p \mathbb{C}^N$:

$$\log (\|\Lambda^p Re^{\lambda P} \Lambda^p U v_1 \wedge \dots \wedge v_p\|) = \frac{1}{2} \log \det_p (v_i^* (e^{\lambda P} U)^* e^{\lambda P} U v_j)_{i,j=1,\dots,p} .$$

Next the identity $\log \det_p = \text{Tr}_p \log$ and an expansion in λ shows equality to

$$\frac{1}{2} \text{Tr}_p \log \left(\mathbf{1}_p + \lambda (v_i^* U^* Q U v_j)_{i,j=1,\dots,p} + \frac{\lambda^2}{2} (v_i^* U^* S U v_j)_{i,j=1,\dots,p} + \mathcal{O}(\lambda^3) \right) .$$

Now an expansion of the logarithm already proves (8). Substituting (8) into (7) one obtains a perturbative formula for the sum of the Lyapunov exponents. Taking the difference of two such sums then allows to deduce a perturbative formula for each Lyapunov exponent γ_p . This procedure and also the above proof of (8) do not provide good error estimates on the dependence of the $\mathcal{O}(\lambda^3)$ term on the matrix size N . Actually, considerably improved bounds can be obtained using the Gram-Schmidt cocycle with values in the upper triangular matrices with

positive diagonal [RS2]. Further improvements in this direction will be provided elsewhere. Here the main focus is on the calculation of the contributions up to order $\mathcal{O}(\lambda^2)$ rather than the error estimate. Let us now come to the results in each of the 10 Cartan classes. Even though the final formulas are all quite compact, it ought to be added that the calculations leading to these formulas are quite tedious. We only provide the main intermediate steps.

2.1 Lyapunov spectrum for Class A

As this is the first case, let us treat it in detail. Due to Section 1.1, the group is $\text{GL}(N, \mathbb{C})$ with compact subgroup $\text{U}(N)$. As there are no specific symmetries, one may choose the unit vectors v_q to be the standard basis vectors e_q having only one non-vanishing entry in the q th component. Then replacing (8) into (7) one sees that one needs to calculate second and fourth moments of the Haar measure on $\text{U}(N)$. The formulas needed are listed in Lemma 1. As $\text{Tr}(\pi_p) = p$, one deduces with some care

$$\sum_{q=1}^p \gamma_q^A = \mathbf{E}_P \left[\lambda \frac{p \text{Tr}(Q)}{2N} + \lambda^2 \frac{p \text{Tr}(S)}{4N} - \lambda^2 \frac{(Np - p^2) \text{Tr}(Q)^2 + (Np^2 - p) \text{Tr}(Q^2)}{4N(N^2 - 1)} \right] + \mathcal{O}(\lambda^3).$$

Now the identity $\text{Tr}(S) = \text{Tr}(Q^2)$ allows to further simplify. Furthermore, taking the difference of $\sum_{q=1}^p \gamma_q$ and $\sum_{q=1}^{p-1} \gamma_q$, one finds

$$\gamma_p^A = \lambda \frac{1}{2N} \mathbf{E}_P \text{Tr}(Q) + \lambda^2 \frac{N+1-2p}{4(N^2-1)} \mathbf{E}_P \left(\text{Tr}(Q^2) - \frac{1}{N} \text{Tr}(Q)^2 \right) + \mathcal{O}(\lambda^3). \quad (9)$$

The main remarkable facts about this formula are the following. To order λ , the Lyapunov spectrum is N -fold degenerate. However, in second order perturbation theory this degeneracy is lifted and, furthermore, the spacings between the Lyapunov exponents are equal. This equidistance is observed in numerous numerical experiments and is here shown to be a consequence of the RPP. Finally, the Lyapunov exponent $\gamma_{[\frac{N+1}{2}]}^A$ is smallest in absolute value. Here $[\cdot]$ denotes the integer part. If N is odd, then $\gamma_{[\frac{N+1}{2}]}^A = 0$ if P is centered. However, this vanishing is lifted by higher order terms, just as a non-centered P would not lead to a non-vanishing Lyapunov exponent. Hence the vanishing is not symmetry enforced, as in Classes DIII and D with odd N .

2.2 Lyapunov spectrum for Class AI

Here all matrices are real as $\mathcal{G}^{\text{AI}} = \text{GL}(N, \mathbb{R})$. Thus also the initial vectors v_1, \dots, v_p should be chosen real. Thus one can again choose $v_q = e_q$. Then one is lead to an analogous calculation as in Class A, but according to the RPP the averages now have to be taken over the orthogonal group instead of the unitary group. The corresponding formulas are collected in Lemma 2. After some algebra one finds

$$\gamma_p^{\text{AI}} = \lambda \frac{1}{2N} \mathbf{E}_P \text{Tr}(Q) + \lambda^2 \frac{N+1-2p}{4(N-1)(N+2)} \mathbf{E}_P \left(\text{Tr}(Q^2) - \frac{1}{N} \text{Tr}(Q)^2 \right) + \mathcal{O}(\lambda^3).$$

The same comments as in Class A apply.

2.3 Lyapunov spectrum for Class AII

Here the matrices $T = Re^{\lambda P}$ are in the group $\mathcal{G}^{\text{AII}} = \text{U}^*(2N)$ and averages have to be taken over the symplectic group $\mathcal{U}^{\text{AII}} = \text{SP}(2N)$. However, some further care is needed with the choice of the v_1, \dots, v_p . Indeed, the Lyapunov exponents are the scaling exponents of the singular values $T_n \cdots T_1$, thus the eigenvalues of the positive matrix $(T_n \cdots T_1)^* T_n \cdots T_1$ in the group $\text{U}^*(2N)$. As explained in Section 1.3, these eigenvalues are twice degenerate and the eigenvectors are pairs $v, I\bar{v}$. Thus also the Lyapunov spectrum is twice degenerate. If one chooses $v_1 = e_1$ and this leads to the largest singular value and thus largest Lyapunov exponent in (7), then one has to choose $v_2 = Ie_1$ so that indeed the second Lyapunov exponent is equal to the first one, hence establishing the degeneracy just explained. Arguing similarly for the other Lyapunov exponents, we therefore set

$$v_{2q-1} = e_q, \quad v_{2q} = I \overline{v_{2q-1}} = I e_q = e_{q+N}.$$

Then the projections $\pi_p = \sum_{q=1}^p v_q v_q^* = (\pi_p)^t$ are of the form

$$\pi_p = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}, \quad b = \sum_{q=1}^{j+\delta} e'_q (e'_q)^*, \quad c = \sum_{q=1}^j e'_q (e'_q)^*, \quad (10)$$

where $p = 2j + \delta$ with $\delta \in \{0, 1\}$ and e'_q denotes again the standard basis vectors of \mathbb{C}^N . Note that both b and c are $N \times N$ matrices. They satisfy $bc = cb = c$ as well as $\text{Tr}(b) = j + \delta$ and $\text{Tr}(c) = j$. This implies

$$I^* \pi_{2j+\delta} I \pi_{2j+\delta} = \pi_{2j}.$$

Furthermore needed is the formula $I^* Q^t I Q = I^* \bar{Q} I Q = Q^2$. With these identities at hand, a careful use of Lemma 3 leads to

$$\begin{aligned} \sum_{q=1}^p \gamma_q^{\text{AII}} &= \mathbf{E}_P \left[\lambda \frac{p \text{Tr}(Q)}{4N} + \lambda^2 \frac{p \text{Tr}(S)}{8N} \right. \\ &\quad \left. - \lambda^2 \frac{(2Np + \delta - p^2) \text{Tr}(Q)^2 + (2Np^2 - 2p - 2Np + 2N\delta) \text{Tr}(Q^2)}{16N(N-1)(2N+1)} \right] + \mathcal{O}(\lambda^3), \end{aligned}$$

where $\delta = \frac{1}{2}(1 - (-1)^p)$ as above. Now again $\text{Tr}(S) = \text{Tr}(Q^2)$ allows to somewhat simplify. Then taking differences a further calculation shows

$$\gamma_p^{\text{AII}} = \lambda \frac{1}{4N} \mathbf{E}_P \text{Tr}(Q) + \lambda^2 \frac{2N+1-2p+(-1)^p}{8(N-1)(2N+1)} \mathbf{E}_P \left(\text{Tr}(Q^2) - \frac{1}{2N} \text{Tr}(Q)^2 \right) + \mathcal{O}(\lambda^3),$$

where $p = 1, \dots, 2N$. Besides the comments made in Class A, let us note that this formula respects the double degeneracy of the Lyapunov spectrum resulting from Kramers' degeneracy.

2.4 Lyapunov spectrum for Classes AIII

The reflection symmetry of the spectrum of matrices in $\mathcal{G}^{\text{AIII}} = \text{U}(N, N)$ first of all implies that the Lyapunov spectrum satisfies $\gamma_p^{\text{AIII}} = -\gamma_{2N+1-p}^{\text{AIII}}$. Hence it is sufficient to calculate the

non-negative Lyapunov exponents. In principle, one can proceed as above, but it is important to choose the vectors v_q J -isotropic because the Lyapunov spectrum studies the scaling of the singular values of products in $U(N, N)$, which are eigenvalues of positive matrices in $U(N, N)$ so that their eigenvectors are always isotropic. Therefore we choose

$$v_q = \frac{1}{\sqrt{2}}(e_q + e_{q+N}). \quad (11)$$

It follows that $\pi_p = \frac{1}{2} \begin{pmatrix} d & d \\ d & d \end{pmatrix}$ where $d = \sum_{q=1}^p e'_q (e'_q)^*$ is an $N \times N$ matrix built from the standard basis e'_q in \mathbb{C}^N . Now the average over $\mathcal{U}^{\text{AIII}} = U(N) \times U(N)$ can be calculated with Lemma 1(i). Furthermore, the relation $QJ + JQ = 0$ implies that $Q = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ is off-diagonal so that, in particular, $\text{Tr}(Q) = 0$. With the notations $\Pi_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\Pi_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ one gets after carrying out the averages over the quadratic terms of U :

$$\begin{aligned} \sum_{q=1}^p \gamma_q^{\text{AIII}} &= \mathbf{E}_P \mathbf{E}_V \mathbf{E}_W \left[\frac{\lambda^2}{4} \frac{1}{2N} (\text{Tr}(S\Pi_+) \text{Tr}(\pi_p \Pi_+) + \text{Tr}(S\Pi_-) \text{Tr}(\pi_p \Pi_-)) \right. \\ &\quad \left. - \frac{\lambda^2}{16} (\text{Tr}(V^* a W d W^* a^* V d) + \text{Tr}(W^* a^* V d V^* a W d)) \right] + \mathcal{O}(\lambda^3), \end{aligned}$$

so that $\text{Tr}(\pi_p \Pi_{\pm}) = \text{Tr}(d) = p$, $\text{Tr}(S) = 2 \text{Tr}(a^* a) = \text{Tr}(Q^2)$ leads to

$$\sum_{q=1}^p \gamma_q^{\text{AIII}} = \frac{\lambda^2}{8} \left(\frac{p}{N} - \frac{p^2}{2N^2} \right) \mathbf{E}_P \text{Tr}(Q^2) + \mathcal{O}(\lambda^3).$$

Taking differences therefore shows

$$\gamma_p^{\text{AIII}} = \lambda^2 \frac{N - p + \frac{1}{2}}{8 N^2} \mathbf{E}_P \text{Tr}(Q^2) + \mathcal{O}(\lambda^3), \quad p = 1, \dots, N, \quad (12)$$

which was already derived in [RS2]. One important feature of this formula is that the smallest non-negative Lyapunov exponent γ_N^{AIII} is strictly positive. This does not hold if one considers random products in $U(N, M)$ with $N < M$. Indeed, then there are $M - N$ vanishing Lyapunov exponents. With some care one can transpose the above calculation to show that the remainder of the Lyapunov spectrum is again given by (12), but in the denominator N^2 is replaced by NM .

2.5 Lyapunov spectrum for Classes CI

In Class CI $\mathcal{G}^{\text{CI}} \cong \text{SP}(2N, \mathbb{R})$ and $\mathcal{U}^{\text{CI}} \cong U(N)$. This was also dealt with in [RS2] and the result is

$$\gamma_p^{\text{CI}} = \lambda^2 \frac{N - p + 1}{8 N(N+1)} \mathbf{E}_P \text{Tr}(Q^2) + \mathcal{O}(\lambda^3), \quad p = 1, \dots, N.$$

Then the other half of the Lyapunov spectrum is again given by $\gamma_p^{\text{CI}} = -\gamma_{2N+1-p}^{\text{CI}}$. The proof is a mixture of Class AIII and DIII below.

2.6 Lyapunov spectrum for Class DIII

In Class DIII the matrices are in the group $\mathcal{G}^{\text{AIII}} = \text{U}^*(2N) \cap \text{U}(N, N) \cong \text{SO}^*(2N)$ so that the Lyapunov spectrum has both the double Kramers' degeneracy and the symmetry around 0. For odd N , this implies that there must be a twice degenerate vanishing Lyapunov exponent. This vanishing is symmetry enforced and thus must also hold to all orders of perturbation theory. The implication of a vanishing Lyapunov exponent are discussed in Section 5. Now let us come to the perturbative calculation of the Lyapunov spectrum. For the choice of the orthonormal basis v_q two criteria have to be satisfied. First of all, the eigenvectors have to be J -isotropic and pairwise J -orthogonal for different eigenvalues as in the case of $\mathcal{G}^{\text{AIII}}$, and second of all, they should be chosen in pairs as in the case of \mathcal{G}^{AII} . Therefore, we set

$$v_{2q-1} = \frac{1}{\sqrt{2}}(e_q + e_{q+N}), \quad v_{2q} = I \overline{v_{2q-1}} = \frac{1}{\sqrt{2}}(e_{q+N} - e_q). \quad (13)$$

Note that both these vectors are J -isotropic, but they are not J -orthogonal (which is not required because they correspond to eigenvectors of the same eigenvalue). However, each is J -orthogonal to all other basis vectors except for his pair partner. Then $\pi_p = \sum_{q=1}^p v_q(v_q)^*$ is precisely the same as in (10). Now the average over the group $\mathcal{U}^{\text{DIII}} \cong \text{U}(N)$ in (7) is carried out using the formulas of Lemma 4. Note that $QJ + JQ = 0$ and $I^* \overline{Q} I = Q$ imply that $Q = Q^* = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ is off-diagonal with $a^t = -a$, so that Lemma 4(iii) applies. Also $\text{Tr}(Q) = 0$. Furthermore, π_p is also of the form required in Lemma 4(ii) and (iii). With the notation Π_{\pm} as in Lemma 4 and writing again $p = 2j + \delta$ with $\delta \in \{0, 1\}$,

$$\sum_{q=1}^p \gamma_q^{\text{DIII}} = \frac{\lambda^2}{4} \mathbf{E}_P \left[\frac{j+\delta}{N} \text{Tr}(\Pi_+ S) + \frac{j}{N} \text{Tr}(\Pi_- S) - \frac{(j+\delta)j-j}{N(N-1)} \text{Tr}(Q^2) \right] + \mathcal{O}(\lambda^3).$$

As one also has $I^* \overline{S} I = S$ and $I^* \Pi_+ I = \Pi_-$, it follows that $\text{Tr}(\Pi_- S) = \text{Tr}(\Pi_+ S) = \frac{1}{2} \text{Tr}(Q^2)$. Thus

$$\sum_{q=1}^p \gamma_q^{\text{DIII}} = \frac{\lambda^2}{16} \left[\frac{2p}{N} - \frac{p^2 - 2p + \delta}{N(N-1)} \right] \mathbf{E}_P \text{Tr}(Q^2) + \mathcal{O}(\lambda^3).$$

From this one deduces

$$\gamma_p^{\text{DIII}} = \lambda^2 \frac{N - p + \frac{1}{2} + \frac{1}{2}(-1)^p}{8N(N-1)} \mathbf{E}_P \text{Tr}(Q^2) + \mathcal{O}(\lambda^3), \quad p = 1, \dots, N.$$

Again the other half of the Lyapunov spectrum is given by $\gamma_p^{\text{DIII}} = -\gamma_{2N+1-p}^{\text{DIII}}$. As already pointed out above, the case of even N was already treated in [RS2].

It is remarkable that the smallest non-negative Lyapunov exponents γ_N^{AIII} , γ_N^{CI} and γ_N^{DIII} for even N are strictly positive and satisfy $2\gamma_N^{\text{AIII}} = \gamma_N^{\text{CI}} + \mathcal{O}(N^{-1})$ and $2\gamma_N^{\text{AIII}} = \gamma_N^{\text{DIII}} + \mathcal{O}(N^{-1})$, the latter again in the case of even N . It reflects the change of the inverse localization length under breaking of time-reversal symmetry, see the discussion in [RS2].

2.7 Lyapunov spectrum for Class BDI

Because the spectrum of a self-adjoint matrix in $\mathcal{G}^{\text{BDI}} = \text{O}(N, N)$ comes in pairs λ, λ^{-1} , the Lyapunov spectrum always satisfies $\gamma_p^{\text{BDI}} = -\gamma_{2N-p}^{\text{BDI}}$. Hence it is again sufficient to calculate the first half of the spectrum. As $\text{O}(N, N) \subset \text{U}(N, N)$, one has to choose the vectors v_q as in (11). These vectors are already chosen to be real which would be the second requirement in the present case. Now both π_p and Q are as in Class AIII, $\pi_p = \frac{1}{2} \begin{pmatrix} d & d \\ d & d \end{pmatrix}$ where $d = \sum_{q=1}^p e'_q (e'_q)^*$ and $Q = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$. The only supplementary property is that a is real. Also the calculation of the Lyapunov spectrum remains the same as in Class AIII, except that the average is now taken w.r.t. the Haar measure on $\text{O}(N)$ instead of $\text{U}(N)$. As only the second moments enter and they are the same by Lemma 2(i) and Lemma 1(i), also the final result of the calculation is the same:

$$\gamma_p^{\text{BDI}} = \lambda^2 \frac{N-p+\frac{1}{2}}{8N^2} \mathbf{E}_P \text{Tr}(Q^2) + \mathcal{O}(\lambda^3), \quad p = 1, \dots, N.$$

If one considers the case of $\text{O}(N, M)$, the same comments as at the end of Section 2.3 apply.

2.8 Lyapunov spectrum for Class CII

As already stressed in Section 1.8, elements of both groups \mathcal{G}^{CII} and $\mathcal{G}^{\text{DIII}}$ have Kramers' degeneracy and reflection symmetry, so that the Lyapunov spectrum is twice degenerate and symmetric around 0. However, the symmetry leading to Kramers' degeneracy is different and therefore instead of (13) one sets for $q = 1, \dots, N$

$$v_{2q-1} = \frac{1}{\sqrt{2}} (e_q + e_{q+2N}), \quad v_{2q} = J \otimes I \overline{v_{2q-1}} = \frac{1}{\sqrt{2}} (e_{q+N} - e_{q+3N}).$$

From this one deduces

$$\pi_p = \begin{pmatrix} e & f \\ f & e \end{pmatrix}, \quad e = \frac{1}{2} \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}, \quad f = \frac{1}{2} \begin{pmatrix} b & 0 \\ 0 & -c \end{pmatrix},$$

with $N \times N$ matrices b and c defined in (10). Thus $IfIf = \frac{1}{4} \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$. In particular, for $p = 2j + \delta$ one has $2 \text{Tr}(e) = p$ and $2 \text{Tr}(IfIf) = j$. Furthermore, $Q = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ with $I\bar{a}I = a$ (note the sign which means that a is not a quaternion matrix, but $\imath a$ is). Now from Lemma 5 one deduces

$$\begin{aligned} \sum_{q=1}^p \gamma_q^{\text{CII}} &= \frac{\lambda^2}{4} \mathbf{E}_P \left[\frac{1}{2N} (\text{Tr}(\Pi_+ S) \text{Tr}(\Pi_+ \pi_p) + \text{Tr}(\Pi_- S) \text{Tr}(\Pi_- \pi_p)) \right. \\ &\quad \left. - \frac{1}{4N^2} \text{Tr}(Q^2) (\text{Tr}(e)^2 + \text{Tr}(I^* f I f)) \right] + \mathcal{O}(\lambda^3), \end{aligned}$$

Replacing thus shows

$$\sum_{q=1}^p \gamma_q^{\text{CII}} = \frac{\lambda^2}{4} \left[\frac{p}{4N} - \frac{1}{4N^2} \left(\frac{p^2}{4} - \frac{j}{2} \right) \right] \mathbf{E}_P \text{Tr}(Q^2) + \mathcal{O}(\lambda^3).$$

Therefore

$$\gamma_p^{\text{CII}} = \frac{\lambda^2}{8} \frac{2N - p + 1 + \frac{1}{2}(-1)^p}{4N^2} \mathbf{E}_P \text{Tr}(Q^2) + \mathcal{O}(\lambda^3), \quad p = 1, \dots, 2N.$$

2.9 Lyapunov spectrum for Class D

The reflection symmetry of the spectrum of a positive matrix in $\mathcal{G}^D = O(N, \mathbb{C})$ discussed in Section 1.9 implies again that the Lyapunov spectrum always satisfies $\gamma_p^D = -\gamma_{N-p}^D$. Therefore, if N is odd, there is one vanishing Lyapunov exponent $\gamma_{\frac{N+1}{2}}^D = 0$. This vanishing is hence symmetry enforced. For the calculation of the Lyapunov spectrum, one starts again from (8) and (7). According the RPP the average now has to be calculated on the group $\mathcal{U}^D = O(N)$, thus pends on the formulas in Lemma 2. Furthermore, one uses that $Q = Q^* \in \mathfrak{o}(N, \mathbb{C})$ satisfies $Q = -Q^t = -\overline{Q}$ so that, in particular, $\text{Tr}(Q) = 0$ and $\text{Tr}(Q^t Q) = -\text{Tr}(Q^2)$. From this and $\text{Tr}(S) = \text{Tr}(Q^2)$ one deduces after some algebra

$$\sum_{q=1}^p \gamma_q^D = \frac{\lambda^2}{4} \left[\frac{p}{N} - \frac{p^2 - p}{N(N-1)} \right] \mathbf{E}_P \text{Tr}(Q^2) - \frac{\lambda^2}{4} \frac{Np - p^2}{N(N-1)(N+2)} \mathbf{E}_P \text{Tr}(Q)^2 + \mathcal{O}(\lambda^3),$$

which implies

$$\gamma_p^D = \frac{\lambda^2}{4} \frac{N+1-2p}{N(N-1)} \mathbf{E}_P \left[\text{Tr}(Q^2) - \frac{1}{N+2} \text{Tr}(Q)^2 \right] + \mathcal{O}(\lambda^3), \quad p = 1, \dots, N.$$

Hence in Class D there is never a term of order $\mathcal{O}(\lambda)$, and the perturbative formula indeed respects $\gamma_{\frac{N+1}{2}}^D = 0$.

2.10 Lyapunov spectrum for Class C

Similar as in Class D the Lyapunov spectrum satisfies $\gamma_p^C = -\gamma_{2N-p}^C$, but as $2N$ is even, this symmetry enforces no vanishing of a Lyapunov exponent. As preparation for the calculation of the Lyapunov spectrum, let us note that $Q = Q^* \in \mathfrak{sp}(2N, \mathbb{C})$ satisfies $Q = -I^* Q^t I = -I^* \overline{Q} I$ and thus $\text{Tr}(Q) = 0$ and $\text{Tr}(I^* Q^t I Q) = -\text{Tr}(Q^2)$. By the RPP, the average in (7) is calculated over the group $\mathcal{U}^C = \text{SP}(2N)$, based Lemma 3. One finds, first for $p \leq N$,

$$\sum_{q=1}^p \gamma_q^C = \frac{\lambda^2}{4} \mathbf{E}_P \left[\frac{p}{2N} \text{Tr}(S) - \frac{p^2}{2N(2N+1)} \text{Tr}(Q^2) \right] + \mathcal{O}(\lambda^3),$$

and therefore

$$\gamma_p^C = \frac{\lambda^2}{4} \frac{N+1-p}{N(2N+1)} \mathbf{E}_P \text{Tr}(Q^2) + \mathcal{O}(\lambda^3), \quad p = 1, \dots, N.$$

The second (negative) half of the Lyapunov spectrum is given by the reflection stated above.

3 Symmetries of one-particle Hamiltonians

This section briefly reviews the ten-fold way of Altland and Zirnbauer [AZ], albeit with a classification of the symmetries in the spirit of [SRFL, RSFL]. In this classification the one-particle (first quantized) Hamiltonian H acting on a complex Hilbert space can have three

basic symmetries, the time reversal symmetry (TRS), the particle-hole symmetry (PHS) and the sublattice or chiral symmetry (SLS). The first two of these symmetries can be either even or odd. These symmetries are implemented by real unitaries J , I , K , L and M in the following manner:

$$\text{(even TRS)} \quad J^* \overline{H} J = H, \quad J^2 = \mathbf{1}, \quad (14)$$

$$\text{(odd TRS)} \quad I^* \overline{H} I = H, \quad I^2 = -\mathbf{1}, \quad (15)$$

$$\text{(even PHS)} \quad K^* \overline{H} K = -H, \quad K^2 = \mathbf{1}, \quad (16)$$

$$\text{(odd PHS)} \quad L^* \overline{H} L = -H, \quad L^2 = -\mathbf{1}, \quad (17)$$

$$\text{(SLS)} \quad M^* H M = -H, \quad M^2 = \pm \mathbf{1}. \quad (18)$$

The notational choice of letters J , I , *etc.*, is not yet connected to the symmetries as defined in (1), but will in some cases be so in our examples later on. For instance, it is possible to have $J = \mathbf{1}$ so that the even PHS just reads $\overline{H} = H$.

Let us add a few comments. A given Hamiltonian H can have only one TRS and one PHS symmetry. If one TRS and one PHS is present, then one also has a SLS symmetry. For example, if (15) and (16) hold, then also (18) with $M = K^* I$. This also imposes the sign of the SLS which is why one does not keep track of it. The only case in which this sign may seem interesting is the case where there is only SLS. But in this situation the reality of M is irrelevant and then M may be changed to $\imath M$ which alters the sign. Counting all possibilities to combine the three symmetries, one obtains 10 classes which are listed in Table 1 (1 with no symmetry, 5 with one symmetry and 4 with three symmetries). The table also contains the Cartan classification associated to each such H Class by Altland and Zirnbauer [AZ] in the following manner. Each Cartan class is specified by an involution on a compact Lie algebra. The -1 eigenspace of the involution multiplied with \imath then produces a set of self-adjoint matrices which forms the corresponding H Class of Hamiltonians. See also [HHZ, SRFL, RSFL, SCR, AK] for further insights on this connection.

4 Quasi-one-dimensional model operators

In this section we construct for every H Class in Table 1 a quasi-one-dimensional model operator of Dirac type. If the symmetries are organized using the invariance under representations of the real and complex Clifford algebras [HHZ, Kit, SCR, AK], then there is a nice constructive way to obtain these model operators. Even though this might be quite illuminating, it is not the focus of the present paper where we only restrict ourselves at providing one example for each given class, similar as in [BMSA, BFGM, TBFM]. Our model operator of Dirac type will have a random perturbation and the study of its fundamental solutions (transfer matrices) then leads to one of the classes of random products described in Section 2. The model operators act on the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^{2N})$ or, if necessary, on $L^2(\mathbb{R}, \mathbb{C}^{2N}) \otimes \mathbb{C}^2$, and are of one of the three following forms:

$$H_J = J \imath \partial - \lambda \mathcal{V}, \quad H_I = I \partial - \lambda \mathcal{V}, \quad H_K = K \imath \partial - \lambda \mathcal{V}, \quad (19)$$

H Class	TRS	PHS	SLS	\mathcal{G} Class	\mathcal{G}	$\mathcal{U} \cong$	μ
A	0	0	0	AIII	$U(N, M)$	$U(N) \times U(M)$	1
AI	+1	0	0	CI	$SP(2N, \mathbb{R})$	$U(N)$	1
AII	-1	0	0	DIII	$SO^*(2N)$	$U(N)$	2
AIII	0	0	1	A	$GL(N, \mathbb{C})$	$U(N)$	1(2)
BDI	+1	+1	1	AI	$GL(N, \mathbb{R})$	$O(N)$	1(2)
CII	-1	-1	1	AII	$U^*(2N)$	$SP(2N)$	2(4)
D	0	+1	0	BDI	$O(N, M)$	$O(N) \times O(M)$	1
C	0	-1	0	CII	$SP(2N, 2M)$	$SP(2N) \times SP(2M)$	1
DIII	-1	+1	1	D	$O(N, \mathbb{C})$	$O(N)$	2
CI	+1	-1	1	C	$SP(2N, \mathbb{C})$	$SP(2N)$	2

Table 1: Table with symmetries taken from [TBFM, SRFL] together with the transfer matrix group \mathcal{G} for quasi-one-dimensional operators H and its maximal compact subgroup \mathcal{U} . Furthermore, μ is the symmetry imposed multiplicity of the Lyapunov spectrum given by the spectral multiplicity of self-adjoint elements of \mathcal{G} . The numbers in parenthesis give the multiplicity to lowest order in perturbation theory for centered perturbations.

Here J , I and K are the matrices given in (1) acting on \mathbb{C}^{2N} and the supplementary \mathbb{C}^2 allows to implement a further symmetry (this is only necessary for H Classes C, CI and CII because an odd PHS cannot be implemented in any of the three model Hamiltonians). Furthermore, ∂ is the derivative on $L^2(\mathbb{R}, \mathbb{C}^{2N})$ and $\mathcal{V} = \mathcal{V}^*$ is a matrix-valued potential. It could be a matrix-valued function, but for sake of concreteness let us rather only consider a potential given by a sum of Dirac peaks:

$$\mathcal{V}(x) = \sum_{n \in \mathbb{Z}} \mathcal{V}_n \delta(x - n). \quad (20)$$

The \mathcal{V}_n are independent and identically distributed random matrices. The formal definition of the singular potential is via boundary conditions, namely either $\psi(n+) = e^{\lambda \imath J \mathcal{V}_n} \psi(n-)$ or $\psi(n+) = e^{\lambda \imath V_n} \psi(n-)$ or $\psi(n+) = e^{\lambda \imath K \mathcal{V}_n} \psi(n-)$, depending on which model in (19) is considered. These formulas combined with Weyl-Titchmarsh theory allow to show that H_J , H_I and H_K are well-defined self-adjoint operators (see [SS2] for details).

In H classes A, C and D it is possible to consider $H_G = G \imath \partial - \lambda \mathcal{V}$ acting on $L^2(\mathbb{R}, \mathbb{C}^{N+M})$ where G is defined in (3) and $N < M$. This models a system with a different number of left and right movers and gives an effective description of chiral edge states of quantum Hall systems (conventional, spin and thermal quantum Hall effect respectively), which is reflected by the fact that the Lyapunov spectrum contains at least $M - N$ vanishing Lyapunov exponents. These systems will be discussed briefly in Section 5.

For each real energy E , the fundamental solutions $T^E(x, y)$ of the Schrödinger equation $HT^E(x, y) = ET^E(x, y)$ which are right continuous in both x and y and satisfy the initial condition $T^E(y, y) = \mathbf{1}$ are also called the transfer matrices of the system. By the above, they

are given by

$$T^E(n-, 1-) = T_n \cdots T_1, \quad (21)$$

where the matrices T_n for three cases in (19) are respectively given by

$$T_n = e^{\imath EJ} e^{\lambda \imath J \mathcal{V}_n}, \quad T_n = e^{EI} e^{\lambda I \mathcal{V}_n}, \quad T_n = e^{\imath EK} e^{\lambda \imath K \mathcal{V}_n}.$$

These matrices are of the form (6) if one sets $R_n = e^{\imath EJ}$ and $P_n = \imath J \mathcal{V}_n$ in the first case, and correspondingly in the other cases. Now by construction, the transfer matrices are in the following three groups:

$$\mathcal{T}_J = \mathrm{U}(N, N), \quad \mathcal{T}_I = C^* \mathrm{U}(N, N) C, \quad \mathcal{T}_K = C \mathrm{U}(N, N) C^*, \quad (22)$$

where C is the Cayley transform defined in (2). For H Class C, CI and CII one has to replace $\mathrm{U}(N, N)$ by $\mathrm{U}(2N, 2N)$ due to the supplementary \mathbb{C}^2 . Thus it looks like the transfer matrices are always in the Cartan Class AIII, but actually the game in the following subsections is to show how the symmetries of the Hamiltonian (TRS, PHS, SLS) lead to subgroups of \mathcal{T}_J , \mathcal{T}_I or \mathcal{T}_K which then actually cover all Cartan classes. Table 1 collects the results of this correspondence.

4.1 Standard unitary class (H Class A)

Let us choose the model H_J . As there are no further symmetries, the transfer matrix group \mathcal{T}_J^A is equal to $\mathcal{G}^{\mathrm{AIII}} = \mathrm{U}(N, N)$ and therefore the results of Section 2.4 can be directly applied to study the associated Lyapunov spectrum.

4.2 Standard orthogonal class (H Class AI)

Let us again choose the model H_J and then implement the even TRS by $K^* \overline{H_J} K = H_J$. Then

$$K^* \overline{T_n} K = K^* e^{-\imath EJ} e^{-\lambda \imath J \overline{\mathcal{V}_n}} K = e^{\imath EJ} e^{\lambda \imath J K^* \overline{\mathcal{V}_n} K} = T_n,$$

shows that $\mathcal{T}_J^{\mathrm{AI}} = \mathcal{G}^{\mathrm{CI}} = C \mathrm{SP}(2N, \mathbb{R}) C^*$ and Section 2.5 applies. Alternatively, one can choose H_I and then implement the even TRS by $\overline{H_I} = H_I$. Then $\mathcal{T}_I^{\mathrm{AI}} = \mathrm{SP}(2N, \mathbb{R})$.

4.3 Standard symplectic class (H Class AII)

Starting again from the model H_J , the odd TRS is now $I^* \overline{H_J} I = H_J$. By a similar calculation as above one finds that $\mathcal{T}_J^{\mathrm{AII}} = \mathcal{G}^{\mathrm{DIII}} = C \mathrm{SO}^*(2N) C^*$. Thus Section 2.6 applies.

4.4 Chiral unitary class (H Class AIII)

When there is a chiral symmetry, it is good to work with the model operator H_I and implement the SLS by $J^* H_I J = -H_I$. The reason is that this implies $J^* T_n J = T_n$, which combined with $T_n^* I T_n = I$ shows that T_n is in the group

$$\mathcal{T}_I^{\mathrm{AIII}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^* \end{pmatrix} \middle| A \in \mathrm{GL}(N, \mathbb{C}) \right\}. \quad (23)$$

Therefore $\mathcal{T}_I^{\text{AIII}} \cong \mathcal{G}^{\text{A}}$ and the results of Section 2.1 for Class A apply. The formula in Section 2.1 gives the Lyapunov spectrum of the upper component in $\mathcal{T}_I^{\text{AIII}}$. The exponents of the lower component are then given exactly by the negative of the upper component. In this manner one recovers the symmetry of the Lyapunov spectrum which has to hold for the group $\mathcal{T}_I^{\text{AIII}} \subset \text{U}(N, N)$. It is worth mentioning the following implication of the comments at the end of Section 2.1. If the potentials \mathcal{V}_n are centered with an distribution that is also even, then already the Lyapunov spectrum of the upper component given in (9) is symmetric around 0 up to terms of order $\mathcal{O}(\lambda^4)$. Therefore, the Lyapunov spectrum in $\mathcal{T}_I^{\text{AIII}}$ has a double degeneracy up to terms of order $\mathcal{O}(\lambda^4)$. If, moreover, N is odd, then there are two Lyapunov exponents that vanish to second order perturbation theory and the first non-vanishing contribution appears only at order $\mathcal{O}(\lambda^4)$. This leads to a very large localization length in such systems. The same comments apply to the other two chiral classes.

4.5 Chiral orthogonal class (H Class BDI)

Everything of the last section transposes, but, moreover, one has the even TRS $\overline{H}_I = H_I$. Combined one also obtains an even PHS $J^* \overline{H}_I J = -H_I$. The transfer matrices are now, on top of being in $\mathcal{T}_I^{\text{AIII}}$, real as in Section 4.2 and therefore

$$\mathcal{T}_I^{\text{BDI}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^* \end{pmatrix} \middle| A \in \text{GL}(N, \mathbb{R}) \right\} \cong \mathcal{G}^{\text{AI}}.$$

Thus the results of Section 2.2 apply, as well as the comments made in Section 4.4.

4.6 Chiral symplectic class (H Class CII)

As in the other two chiral classes, the Hamiltonian is H_I with SLS $J^* H_I J = -H_I$, but now it acts on $L^2(\mathbb{R}, \mathbb{C}^{2N}) \otimes \mathbb{C}^2$ and the supplementary \mathbb{C}^2 is used to implement the odd TRS by $\mathbf{1} \otimes I^* \overline{H}_I \mathbf{1} \otimes I = H_I$ (note that here the $I \otimes \mathbf{1}$ in the definition of H_I and $\mathbf{1} \otimes I$ in the odd TRS are in different gradings). As usual the two symmetries together imply another one, namely an odd PHS $J \otimes I^* \overline{H}_I J \otimes I = -H_I$. One deduces that the transfer matrices satisfy $\mathbf{1} \otimes I^* \overline{T}_n \mathbf{1} \otimes I = T_n$ so that

$$\mathcal{T}_I^{\text{CII}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^* \end{pmatrix} \middle| A \in \text{U}^*(2N) \right\} \cong \mathcal{G}^{\text{AII}}.$$

Thus the results of Section 2.3 apply, which lead in particular to Kramer's degeneracy, which leads even to a fourfold degeneracy up to order $\mathcal{O}(\lambda^4)$ if the distribution of \mathcal{V}_n is even.

4.7 Even PHS without TRS (H Class D)

Here let us choose the model H_J and implement the even PHS simply by $\overline{H}_J = -H_J$. This implies for the transfer matrices that $\overline{T}_n = T_n$, which leads to the following subgroup of \mathcal{T}_J^{A} :

$$\mathcal{T}_J^{\text{D}} = \{ T \in \text{GL}(2N, \mathbb{C}) \mid T^* J T = J, \overline{T} = T \} = \text{O}(N, N) = \mathcal{G}^{\text{BDI}}.$$

Thus the formulas of Section 2.7 apply.

4.8 Odd PHS class without TRS (H Class C)

The odd PHS cannot be implemented in any of the three model Hamiltonians, unless they are tensorized with an extra \mathbb{C}^2 . Having done that, let us choose the model H_J . Then the odd PHS is given by $J \otimes I^* \overline{H_J} J \otimes I = -H_J$. From this one deduces $J \otimes I^* \overline{T_n} J \otimes I = T_n$ so that the transfer matrix group is

$$\mathcal{T}_J^C = \{T \in \text{GL}(4N, \mathbb{C}) \mid T^* J \otimes \mathbf{1} T = J \otimes \mathbf{1}, J \otimes I^* \overline{T} J \otimes I = T\} = \mathcal{G}^{\text{CI}}.$$

Thus the Lyapunov spectrum can be calculated as in Section 2.8.

4.9 Even PHS with odd TRS (H Class DIII)

In Class DIII let us use the model operator H_K and implement the odd TRS by $I^* \overline{H_K} I = H_K$ and the even PHS by $K^* \overline{H_K} K = -H_K$. Then one readily checks that the transfer matrices are in the following group:

$$\mathcal{T}_K^{\text{DIII}} = \{T \in \text{GL}(2N, \mathbb{C}) \mid T^* K T = K, I^* \overline{T} I = T, K^* \overline{T} K = T\}.$$

Writing out the latter relations implies (in particular, first deduce and then use $J^* T J = T$)

$$\mathcal{T}_K^{\text{DIII}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^* \end{pmatrix} \mid A \in \text{O}(N, \mathbb{C}) \right\} \cong \mathcal{G}^{\text{D}}.$$

Thus the results of Section 2.9 apply. In particular, the Lyapunov spectrum of H_K is always twice degenerate (because the spectrum in $\text{O}(N, \mathbb{C})$ is symmetric) and there is always a twice degenerate vanishing Lyapunov exponent if N is odd.

4.10 Odd PHS and even TRS (H Class CI)

Again the odd PHS enforces the use of a supplementary grading, so the Hilbert space is $L^2(\mathbb{R}, \mathbb{C}^{2N}) \otimes \mathbb{C}^2$. Let us choose the model operator H_K and implement the even TRS by $I \otimes I \overline{H_K} I \otimes I = H_K$ and the odd PHS $K \otimes I^* \overline{H_K} K \otimes I = -H_K$. Then also the SLS $J \otimes \mathbf{1} H_K J \otimes \mathbf{1} = -H_K$ holds. Next one verifies that the transfer matrices are in the following group:

$$\mathcal{T}_K^{\text{CI}} = \{T \in \text{GL}(4N, \mathbb{C}) \mid T^* K \otimes \mathbf{1} T = K \otimes \mathbf{1}, K \otimes I^* \overline{T} K \otimes I = T, J \otimes \mathbf{1} T J \otimes \mathbf{1} = T\}.$$

After some algebra, the latter relations show

$$\mathcal{T}_K^{\text{CI}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^* \end{pmatrix} \mid A \in \text{SP}(2N, \mathbb{C}) \right\} \cong \mathcal{G}^{\text{C}}.$$

Hence the results of Section 2.10 apply. In particular, the Lyapunov spectrum of H_K is always twice degenerate (because that in $\text{SP}(2N, \mathbb{C})$ is symmetric).

5 Symmetry enforced delocalization

The paper [SS2] studies the model Hamiltonian H_I of (19) in the symmetry class AII and, in particular, the case where the channel number N is odd. As explained in Section 4.3 there are at least two vanishing Lyapunov exponents in this case, and if a certain coupling hypothesis holds, there are exactly two vanishing Lyapunov exponents, just as in the RPP. By Kotani theory the two vanishing Lyapunov exponents imply that the spectrum of H has almost surely an absolutely continuous component of multiplicity 2. Furthermore, it was possible to show in [SS2] that almost surely there is no singular spectrum provided the distribution of the \mathcal{V}_n is absolutely continuous. All these results actually transpose to H Class DIII in the case where N is odd. No details of proof are provided here because there is really no essential difference.

Finally let us discuss systems with a different number of left and right movers which are relevant for the modeling of the edge modes in various quantum Hall systems.

$$H_G = G \iota \partial + \mathcal{V},$$

acting on $L^2(\mathbb{R}, \mathbb{C}^{N+M})$ where $N \leq M$ are integers. If H satisfies a TRS or a SLS, then $N = M$ because otherwise the kinetic part is not isospectral to its negative. But in the remaining H Classes A, C and D it is permitted to have $M > N$, see Table 1. It follows that the transfer matrices are in the group $U(N, M)$. For the H classes C and D they are actually in subgroups $O(N, M)$ and $SP(N/2, M/2)$, where N and M have to be even in the latter case. Now the spectral theory of a self-adjoint $T \in U(N, M)$ shows that T has at least $M - N$ eigenvalues 1, see Section 1.4. Thus H_G must have $M - N$ vanishing Lyapunov exponents. Under a coupling hypothesis on the random potential there are no other vanishing Lyapunov exponents. Again it follows from Kotani theory alluded to above that these Hamiltonians have almost surely an absolutely continuous spectrum of multiplicity $M - N$. Again one can show that this absolutely continuous spectrum is almost surely pure if the distribution of the \mathcal{V}_n is absolutely continuous.

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A Moments for the Haar measures over compact groups

Section A.1 of this appendix collects formulas for certain averages over the compact classical groups that are used in the main part of this paper. These formulas can be derived rather easily from results scattered over the literature, and those involving averages over the unitary group were already listed in [RS2]. Nevertheless, it is emphasized in the remaining sections of the appendix that all these formulas follow from a useful calculus involving the so-called Weingarten function [Wei, CS, CSt] which may be viewed as a systematic approach to invariant theoretic arguments in the physics literature on averages, as in Section VIII of [AZ]. To illustrate how invariant theory is applied, Section A.2 deals explicitly with the fourth moments over the

symplectic group (averages of a product of four matrix entries) and provides the corresponding value of the Weingarten function. This allows in Section A.3 to deduce Lemma 3(ii) which is actually the most complicated of all formulas in Section A.1. All others follow similarly from prior results on the orthogonal group [CS]. Higher moments can also be calculated explicitly, provided that the matrix size is sufficiently large compared to the order of the moment (that is, the number of matrix entries that are being considered). Section A.4 presents the general version of the Weingarten integration formula for the unitary, orthogonal, and symplectic groups. These results are not really used in this paper, but they are definitely of independent interest and are provided for further reference.

A.1 Collection of formulas used in Section 2

Lemma 1 *Let $A, B, C, D \in \text{Mat}(N \times N, \mathbb{C})$. The following holds for averages over $U(N)$:*

$$\begin{aligned}
\text{(i)} \quad & \langle \text{Tr}(U^* A U B) \rangle = \frac{1}{N} \text{Tr}(A) \text{Tr}(B) , \\
\text{(ii)} \quad & \langle \text{Tr}(\bar{U} A U B) \rangle = \frac{1}{N} \text{Tr}(A B^t) , \quad \langle \text{Tr}(U^t A U B) \rangle = 0 , \\
\text{(iii)} \quad & \langle \text{Tr}(U^* A U B U^* C U D) \rangle = \frac{1}{N^2 - 1} \left[\text{Tr}(A) \text{Tr}(C) \text{Tr}(B D) + \text{Tr}(A C) \text{Tr}(B) \text{Tr}(D) \right] \\
& \quad - \frac{1}{N(N^2 - 1)} \left[\text{Tr}(A C) \text{Tr}(B D) + \text{Tr}(A) \text{Tr}(B) \text{Tr}(C) \text{Tr}(D) \right] , \\
\text{(iv)} \quad & \langle \text{Tr}(U^* A U B U^t C \bar{U} D) \rangle = \frac{1}{N^2 - 1} \left[\text{Tr}(A) \text{Tr}(C) \text{Tr}(B D) + \text{Tr}(A C^t) \text{Tr}(B D^t) \right] \\
& \quad - \frac{1}{N(N^2 - 1)} \left[\text{Tr}(A C^t) \text{Tr}(B D) + \text{Tr}(A) \text{Tr}(C) \text{Tr}(B D^t) \right] , \\
\text{(v)} \quad & \langle \text{Tr}(U^* A \bar{U} B U^t C U D) \rangle = \frac{1}{N^2 - 1} \left[\text{Tr}(A C^t) \text{Tr}(B D^t) + \text{Tr}(A C) \text{Tr}(B) \text{Tr}(D) \right] \\
& \quad - \frac{1}{N(N^2 - 1)} \left[\text{Tr}(A C) \text{Tr}(B D^t) + \text{Tr}(A C^t) \text{Tr}(B) \text{Tr}(D) \right] .
\end{aligned}$$

All formulas remain valid if all U 's are replaced by their complex conjugates, e.g.

$$\langle \text{Tr}(U^t A \bar{U} B U^t C \bar{U} D) \rangle = \langle \text{Tr}(U^* A U B U^* C U D) \rangle .$$

Lemma 2 *Let $A, B, C, D \in \text{Mat}(N \times N, \mathbb{C})$. The following holds for averages over $O(N)$:*

$$\begin{aligned}
\text{(i)} \quad & \langle \text{Tr}(O^t A O B) \rangle = \frac{1}{N} \text{Tr}(A) \text{Tr}(B) , \quad \langle \text{Tr}(O A O B) \rangle = \frac{1}{N} \text{Tr}(A B^t) .
\end{aligned}$$

(ii)

$$\begin{aligned}
& \langle \text{Tr}(O^t A O B O^t C O D) \rangle \\
&= \frac{N+1}{N(N-1)(N+2)} \left[\text{Tr}(A)\text{Tr}(C)\text{Tr}(BD) + \text{Tr}(AC^t)\text{Tr}(BD^t) + \text{Tr}(AC)\text{Tr}(B)\text{Tr}(D) \right] \\
&\quad - \frac{1}{N(N-1)(N+2)} \left[\text{Tr}(A)\text{Tr}(C)\text{Tr}(BD^t) + \text{Tr}(A)\text{Tr}(C)\text{Tr}(B)\text{Tr}(D) + \text{Tr}(AC^t)\text{Tr}(BD) \right. \\
&\quad \left. + \text{Tr}(AC^t)\text{Tr}(B)\text{Tr}(D) + \text{Tr}(AC)\text{Tr}(BD) + \text{Tr}(AC)\text{Tr}(BD^t) \right].
\end{aligned}$$

(iii)

$$\begin{aligned}
& \langle \text{Tr}(O^t A O^t B O C O D) \rangle \\
&= \frac{N+1}{N(N-1)(N+2)} \left[\text{Tr}(A^t B C^t D) + \text{Tr}(D C B A) + \text{Tr}(AC)\text{Tr}(B)\text{Tr}(D) \right] \\
&\quad - \frac{1}{N(N-1)(N+2)} \left[\text{Tr}(A^t B C^t D^t) + \text{Tr}(A^t B C^t)\text{Tr}(D) + \text{Tr}(C B A D^t) \right. \\
&\quad \left. + \text{Tr}(C B A)\text{Tr}(D) + \text{Tr}(C A D^t)\text{Tr}(B) + \text{Tr}(C A D^t)\text{Tr}(B) \right].
\end{aligned}$$

Lemma 3 Let $A, B, C, D \in \text{Mat}(2N \times 2N, \mathbb{C})$. The following holds for averages over $\text{SP}(2N)$:

(i) One has

$$\langle \text{Tr}(U^* A U B) \rangle = \frac{1}{2N} \text{Tr}(A) \text{Tr}(B), \quad \langle \text{Tr}(U^t A U B) \rangle = \frac{1}{2N} \text{Tr}(I A) \text{Tr}(I^* B), \quad (24)$$

and

$$\langle \text{Tr}(U A U B) \rangle = \langle \text{Tr}(U^* A U^* B) \rangle = \frac{1}{2N} \text{Tr}(A I B^t I), \quad \langle \text{Tr}(\bar{U} A U B) \rangle = \frac{1}{2N} \text{Tr}(A B^t).$$

(ii)

$$\begin{aligned}
& \langle \text{Tr}(U^* A U B U^* C U D) \rangle \\
&= \frac{1}{4N(N-1)(2N+1)} \left[(2N-1)\text{Tr}(A)\text{Tr}(C)\text{Tr}(BD) + \text{Tr}(A)\text{Tr}(C)\text{Tr}(I^* B^t I D) \right. \\
&\quad - \text{Tr}(A)\text{Tr}(C)\text{Tr}(B)\text{Tr}(D) - \text{Tr}(I^* A^t I C)\text{Tr}(BD) + \text{Tr}(I^* A^t I C)\text{Tr}(B)\text{Tr}(D) \\
&\quad - \text{Tr}(AC)\text{Tr}(BD) - (2N-1)\text{Tr}(I^* A^t I C)\text{Tr}(I^* B^t I D) \\
&\quad \left. - \text{Tr}(AC)\text{Tr}(I^* B^t I D) + (2N-1)\text{Tr}(AC)\text{Tr}(B)\text{Tr}(D) \right].
\end{aligned}$$

Let us close this section by stating two results that follow from the above. Recall from Section 1.5 the special form of $\mathcal{U}^{\text{CI}} = \mathcal{U}^{\text{DIII}} \subset \text{U}(2N)$ which is isomorphic to $\text{U}(N)$. The following lemma presents no general formula for second and fourth moments of this group, but only those moments which are needed in Section 2.6. The proof of the lemma is a calculation based on the identities in Lemma 1.

Lemma 4 Let $A, B \in \text{Mat}(2N \times 2N, \mathbb{C})$. Then the averages over $\mathcal{U}^{\text{CI}} = \mathcal{U}^{\text{DIII}}$ lead to the following:

(i) With $\Pi_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\Pi_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\langle \text{Tr}(U^* A U B) \rangle = \frac{1}{N} \text{Tr}(\Pi_+ A) \text{Tr}(\Pi_+ B) + \frac{1}{N} \text{Tr}(\Pi_- A) \text{Tr}(\Pi_- B) .$$

(ii) Let A and B be of the form

$$A = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix} , \quad B = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} ,$$

where all entries are of size $N \times N$. Then

$$\langle \text{Tr}(U^* A U B U^* A U B) \rangle = \frac{2\text{Tr}(a^* a)(N\text{Tr}(b)\text{Tr}(c) - \text{Tr}(bc^t)) + 2\text{Tr}(\bar{a}a)(N\text{Tr}(bc^t) - \text{Tr}(b)\text{Tr}(c))}{N(N^2 - 1)} .$$

(iii) If, moreover, $a^t = -a$, then

$$\langle \text{Tr}(U^* A U B U^* A U B) \rangle = \frac{\text{Tr}(A^2)(\text{Tr}(b)\text{Tr}(c) - \text{Tr}(bc^t))}{N(N - 1)} .$$

Recall from Section 1.5 the special form of $\mathcal{U}^{\text{CI}} = \text{SP}(2N) \times \text{SP}(2N)$. The following lemma presents no general formula for second and fourth moments of this group, but only those moments which are needed in Section 2.8. The proof of the lemma is a somewhat tedious calculation based on the identities in Lemma 3(i).

Lemma 5 Let $A, B \in \text{Mat}(4N \times 4N, \mathbb{C})$. Then the averages over \mathcal{U}^{CI} lead to the following:

(i) With $\Pi_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\Pi_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ as above,

$$\langle \text{Tr}(U^* A U B) \rangle = \frac{1}{2N} \text{Tr}(\Pi_+ A) \text{Tr}(\Pi_+ B) + \frac{1}{2N} \text{Tr}(\Pi_- A) \text{Tr}(\Pi_- B) .$$

(ii) Let A and B be of the form

$$A = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix} , \quad B = \begin{pmatrix} e & f \\ f & e \end{pmatrix} ,$$

where all entries are of size $2N \times 2N$ and furthermore $I\bar{a}I = a$ and $B^* = B^t = B$. Then

$$\langle \text{Tr}(U^* A U B U^* A U B) \rangle = \frac{\text{Tr}(A^2)(\text{Tr}(e)^2 + \text{Tr}(I^* f I f))}{4N^2} .$$

A.2 The symplectic Weingarten function for fourth moments

As already pointed out, the formulas of Section A.1 can be deduced from [Col, CS] except for averages involving the symplectic group. In the symplectic case a sign factor is missing in [CS]. This was corrected in [CSt], but no explicit value of the Weingarten function was given. This section treats the fourth moment needed for the proof of Lemma 3(ii) in detail. This also indicates how in principle the general results stated in Section A.4 can be proved.

Let e_j , $j = 1, \dots, 2N$, denote the standard basis of the complex vector space $V = \mathbb{C}^{2N}$. For any linear map T on V , the matrix entries are denoted by $T_{i,j} = e_i^* T e_j$. Associated to I defined in (1), let us introduce on V the skew symmetric bilinear form $a(v, w) = v^t I w$ where $v^t = \bar{v}^*$ denotes the transpose of $v \in V$. Then for $i, j = 1, \dots, N$,

$$\begin{aligned} T_{i+N,j} &= -a(e_i, T e_j), & T_{i+N,j+N} &= -a(e_i, T e_{j+N}), \\ T_{i,j} &= a(e_{i+N}, T e_j), & T_{i,j+N} &= a(e_{i+N}, T e_{j+N}). \end{aligned}$$

Thus, for $i_1, j_1, \dots, i_4, j_4 \in \{1, \dots, N\}$ and $\alpha_1, \beta_1, \dots, \alpha_4, \beta_4 \in \{0, 1\}$,

$$\begin{aligned} & \int_{\text{SP}(2N)} dU U_{i_1+\alpha_1 N, j_1+\beta_1 N} U_{i_2+\alpha_2 N, j_2+\beta_2 N} U_{i_3+\alpha_3 N, j_3+\beta_3 N} U_{i_4+\alpha_4 N, j_4+\beta_4 N} \\ &= (-1)^{\alpha_1+\alpha_2+\alpha_3+\alpha_4} a \left(e_{i_1+(1-\alpha_1)N} \otimes \dots \otimes e_{i_4+(1-\alpha_4)N}, \int_{\text{SP}(2N)} dU U_{e_{j_1+\beta_1 N}} \otimes \dots \otimes U_{e_{j_4+\beta_4 N}} \right), \end{aligned} \quad (25)$$

where a is extended to $V^{\otimes 4}$ via

$$a(v_1 \otimes v_2 \otimes v_3 \otimes v_4, w_1 \otimes w_2 \otimes w_3 \otimes w_4) = a(v_1, w_1) a(v_2, w_2) a(v_3, w_3) a(v_4, w_4).$$

Note that this yields a symmetric bilinear form on $V^{\otimes 4}$. By the definition of Haar measure, the integral

$$\int_{\text{SP}(2N)} dU U_{e_{j_1+\beta_1 N}} \otimes \dots \otimes U_{e_{j_4+\beta_4 N}} \quad (26)$$

is a $\text{SP}(2N)$ -invariant in $V^{\otimes 4}$, specifically, it is fixed under the action of $U \in \text{SP}(2N)$ on $V^{\otimes 4}$ which on decomposable tensors is given via $U(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = (U v_1 \otimes U v_2 \otimes U v_3 \otimes U v_4)$. If $N \geq 2$, then by the symplectic case of Weyl's First Fundamental Theorem for tensor invariants [GW, Thm 5.3.3] and the results of Section 3.4 in [Sto], a basis of the (three-dimensional) subspace of invariant tensors in $V^{\otimes 4}$ can be described as follows. Write $\mathbf{p}_1 = \{\{1, 2\}, \{3, 4\}\}$, $\mathbf{p}_2 = \{\{1, 3\}, \{2, 4\}\}$ and $\mathbf{p}_3 = \{\{1, 4\}, \{2, 3\}\}$. These are the pair partitions of the set $\{1, 2, 3, 4\}$. To control the sign factors that arise from the skew symmetry of the form a on V , let us keep track of the natural order on the individual blocks by using the following notation. For $\mathbf{m} \in \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$, let us write $\mathbf{m} = \{(m_1(\mathbf{m}), n_1(\mathbf{m})), (m_2(\mathbf{m}), n_2(\mathbf{m}))\}$, where $\{m_1(\mathbf{m}), n_1(\mathbf{m})\}, \{m_2(\mathbf{m}), n_2(\mathbf{m})\}$ are the blocks of \mathbf{m} , and $m_1(\mathbf{m}) < n_1(\mathbf{m})$, $m_2(\mathbf{m}) < n_2(\mathbf{m})$. To \mathbf{p}_1 is now associated the tensor

$$\theta_1 = \theta_{\mathbf{p}_1} = \sum_{\eta_1, \eta_2=1}^N \sum_{\epsilon_1, \epsilon_2=0}^1 e_{\eta_1+\epsilon_1 N} \otimes e_{\eta_1+(1-\epsilon_1)N} (-1)^{\epsilon_1} \otimes e_{\eta_2+\epsilon_2 N} \otimes e_{\eta_2+(1-\epsilon_2)N} (-1)^{\epsilon_2}.$$

Here, the maps $r \mapsto \eta_r$ and $r \mapsto \epsilon_r$ are constant on the block $\{m_1(\mathbf{p}_1), n_1(\mathbf{p}_1)\}$ as well as on $\{m_2(\mathbf{p}_1), n_2(\mathbf{p}_1)\}$, and to the m component of a block corresponds a vector of the form $e_{\eta_r + \epsilon_r N}$, while to the n component corresponds a vector of the form $(-1)^{\epsilon_r} e_{\eta_r + (1 - \epsilon_r)N}$. Analogously,

$$\begin{aligned}\theta_2 &= \theta_{\mathbf{p}_2} = \sum_{\eta_1, \eta_2=1}^N \sum_{\epsilon_1, \epsilon_2=0}^1 e_{\eta_1 + \epsilon_1 N} \otimes e_{\eta_2 + \epsilon_2 N} \otimes e_{\eta_1 + (1 - \epsilon_1)N} (-1)^{\epsilon_1} \otimes e_{\eta_2 + (1 - \epsilon_2)N} (-1)^{\epsilon_2}, \\ \theta_3 &= \theta_{\mathbf{p}_3} = \sum_{\eta_1, \eta_2=1}^N \sum_{\epsilon_1, \epsilon_2=0}^1 e_{\eta_1 + \epsilon_1 N} \otimes e_{\eta_2 + \epsilon_2 N} \otimes e_{\eta_2 + (1 - \epsilon_2)N} (-1)^{\epsilon_2} \otimes e_{\eta_1 + (1 - \epsilon_1)N} (-1)^{\epsilon_1}.\end{aligned}$$

Then $\{\theta_1, \theta_2, \theta_3\}$ is a basis of the space of symplectic invariants in $V^{\otimes 4}$ and it is a straightforward (tedious) computation to verify that the Gram matrix of the form a with respect to this basis is given by

$$\begin{pmatrix} 4N^2 & 2N & -2N \\ 2N & 4N^2 & 2N \\ -2N & 2N & 4N^2 \end{pmatrix}.$$

As long as $N \geq 2$, this matrix is invertible with inverse

$$\frac{1}{4N(N-1)(2N+1)} \begin{pmatrix} 2N-1 & -1 & 1 \\ -1 & 2N-1 & -1 \\ 1 & -1 & 2N-1 \end{pmatrix}. \quad (27)$$

By definition, the (i, j) -entry of this inverse is the value $\text{Wg}_{\text{SP}(2N)}(\mathbf{p}_i, \mathbf{p}_j)$ of the symplectic Weingarten function. In Section A.4, the general Weingarten function $\text{Wg}_{\text{SP}(2N)}$ is defined by the matrix entries of the inverse of the Gram matrix of a w.r.t. a basis of the invariants, which in turn is in bijection with the pair partitions (of $2k$ points if $2k$ th moments are considered). In order to evaluate (25), one now expresses the invariant (26) in terms of the basis $\{\theta_1, \theta_2, \theta_3\}$. Noting that $a(e_{i_1 + \alpha_1 N} \otimes \dots \otimes e_{i_4 + \alpha_4 N}, \theta_{\mathbf{p}_j})$ vanishes unless $i_{m_r(\mathbf{p}_j)} = i_{n_r(\mathbf{p}_j)}$ and $\alpha_{m_r(\mathbf{p}_j)} = 1 - \alpha_{n_r(\mathbf{p}_j)}$ for $r = 1, 2$, one obtains

$$\begin{aligned} & \int_{\text{SP}(2N)} dU U_{i_1 + \alpha_1 N, j_1 + \beta_1 N} U_{i_2 + \alpha_2 N, j_2 + \beta_2 N} U_{i_3 + \alpha_3 N, j_3 + \beta_3 N} U_{i_4 + \alpha_4 N, j_4 + \beta_4 N} \\ &= \sum_{\mathbf{m}, \mathbf{n} \in \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}} \text{Wg}_{\text{SP}(2N)}(\mathbf{m}, \mathbf{n}) (-1)^{\alpha_{m_1(\mathbf{m})} + \alpha_{m_2(\mathbf{m})} + \beta_{m_1(\mathbf{n})} + \beta_{m_2(\mathbf{n})}} \\ & \quad \delta(i_{m_1(\mathbf{m})}, i_{n_1(\mathbf{m})}) \delta(\alpha_{m_1(\mathbf{m})}, 1 - \alpha_{n_1(\mathbf{m})}) \delta(i_{m_2(\mathbf{m})}, i_{n_2(\mathbf{m})}) \delta(\alpha_{m_2(\mathbf{m})}, 1 - \alpha_{n_2(\mathbf{m})}) \\ & \quad \delta(j_{m_1(\mathbf{n})}, j_{n_1(\mathbf{n})}) \delta(\beta_{m_1(\mathbf{n})}, 1 - \beta_{n_1(\mathbf{n})}) \delta(j_{m_2(\mathbf{n})}, j_{n_2(\mathbf{n})}) \delta(\beta_{m_2(\mathbf{n})}, 1 - \beta_{n_2(\mathbf{n})}). \end{aligned} \quad (28)$$

A.3 Proof of Lemma 3(ii)

Let us begin by writing out the l.h.s. explicitly:

$$\begin{aligned}
& \langle \text{Tr}(IU^t I A U B I U^t I C U D) \rangle \\
&= \sum_{i_1, i_2, \dots, i_8=1}^N \sum_{\alpha_1, \alpha_2, \dots, \alpha_8=0}^1 I_{i_1+\alpha_1 N, i_1+(1-\alpha_1)N} I_{i_2+(1-\alpha_2)N, i_2+\alpha_2 N} A_{i_2+\alpha_2 N, i_3+\alpha_3 N} B_{i_4+\alpha_4 N, i_5+\alpha_5 N} \\
&\quad I_{i_5+\alpha_5 N, i_5+(1-\alpha_5)N} I_{i_6+(1-\alpha_6)N, i_6+\alpha_6 N} C_{i_6+\alpha_6 N, i_7+\alpha_7 N} D_{i_8+\alpha_8 N, i_1+\alpha_1 N} \\
&\quad \langle U_{i_2+(1-\alpha_2)N, i_1+(1-\alpha_1)N} U_{i_3+\alpha_3 N, i_4+\alpha_4 N} U_{i_6+(1-\alpha_6)N, i_5+(1-\alpha_5)N} U_{i_7+\alpha_7 N, i_8+\alpha_8 N} \rangle.
\end{aligned}$$

The next step is to expand this average $\langle \cdot \rangle = \int_{\text{SP}(2N)} dU$ according to (28). Then one interchanges the sum over pairs of pair partitions with the sums over the i and α indices and then determines the contribution of each pair $(\mathbf{p}_i, \mathbf{p}_j)$. Let us illustrate this procedure for the pair $(\mathbf{p}_1, \mathbf{p}_2)$. One reads off from (28) that for a choice of i and α indices to give a non-vanishing contribution, one must have that $i_3 = i_2$, $\alpha_3 = \alpha_2$, $i_7 = i_6$, $\alpha_7 = \alpha_6$, $i_5 = i_1$, $\alpha_5 = 1 - \alpha_1$, $i_8 = i_4$, $\alpha_8 = 1 - \alpha_4$. Furthermore, the sign factor corresponding to each choice of i and α indices is

$$(-1)^{(1-\alpha_2)+(1-\alpha_6)+(1-\alpha_1)+\alpha_4} = (-1)^{\alpha_1+\alpha_2+\alpha_4+(1-\alpha_6)}.$$

Consequently, the coefficient of $\text{Wg}_{\text{SP}(2N)}(\mathbf{p}_1, \mathbf{p}_2)$ in the sum over pairs of pair partitions is

$$\begin{aligned}
& \sum_{i, \alpha} (-1)^{\alpha_1+\alpha_2+\alpha_4+(1-\alpha_6)} I_{i_1+\alpha_1 N, i_1+(1-\alpha_1)N} I_{i_2+(1-\alpha_2)N, i_2+\alpha_2 N} A_{i_2+\alpha_2 N, i_2+\alpha_2 N} B_{i_4+\alpha_4 N, i_1+(1-\alpha_1)N} \\
& \quad I_{i_1+(1-\alpha_1)N, i_1+\alpha_1 N} I_{i_6+(1-\alpha_6)N, i_6+\alpha_6 N} C_{i_6+\alpha_6 N, i_6+\alpha_6 N} D_{i_4+(1-\alpha_4)N, i_1+\alpha_1 N}.
\end{aligned}$$

Observing that $I_{i+\alpha N, i+(1-\alpha)N} = (-1)^{1-\alpha}$ and $I_{i+(1-\alpha)N, i+\alpha N} = (-1)^\alpha$, one can group the factors as follows:

$$\begin{aligned}
& A_{i_2+\alpha_2 N, i_2+\alpha_2 N}, \\
& C_{i_6+\alpha_6 N, i_6+\alpha_6 N}, \\
& B_{i_4+\alpha_4 N, i_1+(1-\alpha_1)N} I_{i_1+(1-\alpha_1)N, i_1+\alpha_1 N} D_{i_4+(1-\alpha_4)N, i_1+\alpha_1 N} (-1)^{\alpha_4} \\
&= B_{i_4+\alpha_4 N, i_1+(1-\alpha_1)N} I_{i_1+(1-\alpha_1)N, i_1+\alpha_1 N} D_{i_4+(1-\alpha_4)N, i_1+\alpha_1 N} I_{i_4+(1-\alpha_4)N, i_4+\alpha_4 N}, \\
& I_{i_1+\alpha_1 N, i_1+(1-\alpha_1)N} (-1)^{\alpha_1} = -1, \\
& I_{i_2+(1-\alpha_2)N, i_2+\alpha_2 N} (-1)^{\alpha_2} = 1, \\
& I_{i_6+(1-\alpha_6)N, i_6+\alpha_6 N} (-1)^{1-\alpha_6} = -1.
\end{aligned}$$

Hence, $\text{Wg}_{\text{SP}(2N)}(\mathbf{p}_1, \mathbf{p}_2)$ comes with the coefficient $\text{Tr}(A) \text{Tr}(C) \text{Tr}(B I D^t I)$. Going in this manner through all nine cases yields

$$\begin{aligned}
& \langle \text{Tr}(U^{-1} A U B U^{-1} C U D) \rangle = \langle \text{Tr}(I U^t I A U B I U^t I C U D) \rangle \\
&= \text{Wg}(\mathbf{p}_1, \mathbf{p}_1) \text{Tr}(A) \text{Tr}(C) \text{Tr}(B D) + \text{Wg}(\mathbf{p}_1, \mathbf{p}_2) \text{Tr}(A) \text{Tr}(C) \text{Tr}(B I D^t I) \\
&\quad - \text{Wg}(\mathbf{p}_1, \mathbf{p}_3) \text{Tr}(A) \text{Tr}(B) \text{Tr}(C) \text{Tr}(D) - \text{Wg}(\mathbf{p}_2, \mathbf{p}_1) \text{Tr}(A I C^t I) \text{Tr}(B D) \\
&\quad - \text{Wg}(\mathbf{p}_2, \mathbf{p}_2) \text{Tr}(A I C^t I) \text{Tr}(B I D^t I) + \text{Wg}(\mathbf{p}_2, \mathbf{p}_3) \text{Tr}(A^t I C I) \text{Tr}(B) \text{Tr}(D) \\
&\quad - \text{Wg}(\mathbf{p}_3, \mathbf{p}_1) \text{Tr}(A C) \text{Tr}(B D) - \text{Wg}(\mathbf{p}_3, \mathbf{p}_2) \text{Tr}(A C) \text{Tr}(B I D^t I) \\
&\quad + \text{Wg}(\mathbf{p}_3, \mathbf{p}_3) \text{Tr}(A C) \text{Tr}(B) \text{Tr}(D),
\end{aligned}$$

and then the claim follows from (27).

A.4 The general Weingarten integration formulas

This appendix recollects the general results on the integration over the orthogonal, symplectic and unitary group as given in (or can be deduced from) [CS], [CSt] and [Col] respectively. The following notation will be used. For $m, n \in \mathbb{N}$ denote by $\mathcal{F}(m, n)$ the set of all maps from $\{1, \dots, m\}$ to $\{1, \dots, n\}$. If \mathbf{m} is a partition of $\{1, \dots, m\}$, then $\mathcal{F}(\mathbf{m}, n)$ denotes the set of those elements of $\mathcal{F}(m, n)$ which are constant on the blocks of \mathbf{m} . Denote by $\mathcal{PP}(k)$ the set of all pair partitions of $\{1, \dots, k\}$. In particular, $\mathcal{PP}(k) = \emptyset$ if k is odd. If $k = 2l$ is even, we write $\mathcal{PP}(k) \ni \mathbf{m} = \{\{m_\nu(\mathbf{m}), n_\nu(\mathbf{m})\} \mid \nu = 1, \dots, l\}$, where the numbering of the blocks is arbitrary, but $m_\nu < n_\nu$ holds for all ν . For maps $\alpha \in \mathcal{F}(k, \{0, 1\})$ from $\{1, \dots, k\}$ to $\{0, 1\}$, let us write $\alpha \in \mathcal{A}(\mathbf{m}, \{0, 1\})$ if for all ν holds $\alpha(m_\nu(\mathbf{m})) = 1 - \alpha(n_\nu(\mathbf{m}))$.

Now let us begin with the orthogonal case. If $N \geq l$, then the space of $O(N)$ -invariants in $V^{\otimes 2l}$ admits a basis $\{\theta_{\mathbf{m}} \mid \mathbf{m} \in \mathcal{PP}(2l)\}$ given by

$$\theta_{\mathbf{m}} = \sum_{\phi \in \mathcal{F}(\mathbf{m}, N)} e_{\phi(1)} \otimes \dots \otimes e_{\phi(2l)},$$

where as above $e_i, i = 1, \dots, N$, is the standard orthonormal basis on $V = \mathbb{C}^N$ which is furnished with the symmetric bilinear form $b(v, w) = v^t w$. The Weingarten function $\text{Wg}_{O(N)}(\mathbf{m}, \mathbf{n})$ is the (\mathbf{m}, \mathbf{n}) -entry of the inverse of the Gram matrix of the extension of b via $b(v_1 \otimes \dots \otimes v_{2l}, w_1 \otimes \dots \otimes w_{2l}) = b(v_1, w_1) \cdot \dots \cdot b(v_{2l}, w_{2l})$ with respect to the basis $\{\theta_{\mathbf{m}} \mid \mathbf{m} \in \mathcal{PP}(2l)\}$. For example, on $\mathcal{PP}(2) \times \mathcal{PP}(2)$ one obtains $\text{Wg}_{O(N)} = \frac{1}{N}$, while on $\mathcal{PP}(4) \times \mathcal{PP}(4)$ the orthogonal Weingarten function $\text{Wg}_{O(N)}$ is given by the matrix entries

$$\frac{1}{N(N-1)(N+2)} \begin{pmatrix} N+1 & -1 & -1 \\ -1 & N+1 & -1 \\ -1 & -1 & N+1 \end{pmatrix}.$$

The general orthogonal Weingarten integration formula now reads as follows.

Proposition 1 *Let $\phi, \psi \in \mathcal{F}(k, N)$. Then*

$$\int_{O(N)} dU \prod_{j=1}^k U_{\phi(j), \psi(j)} = \sum_{\mathbf{m}, \mathbf{n} \in \mathcal{PP}(k)} 1_{\mathcal{F}(\mathbf{m}, n)}(\phi) 1_{\mathcal{F}(\mathbf{n}, n)}(\psi) \text{Wg}_{O(N)}(\mathbf{m}, \mathbf{n}).$$

Next let us consider the symplectic case, hence $V = \mathbb{C}^{2N}$ with a as in Section A.2. If $N \geq l$, then the space of $\text{SP}(2N)$ -invariants in $V^{\otimes 2l}$ admits a basis $\{\theta_{\mathbf{m}} \mid \mathbf{m} \in \mathcal{PP}(2l)\}$, where

$$\theta_{\mathbf{m}} = \sum_{\eta \in \mathcal{F}(l, n)} \sum_{\varepsilon \in \mathcal{F}(l, \{0, 1\})} \otimes_{j=1}^{2l} v(\eta, \varepsilon, j),$$

with

$$v(\eta, \epsilon, j) = \begin{cases} e_{\nu(\eta+n\epsilon)} , & \text{if } j = m_\nu , \\ e_{\nu(\eta+n(1-\epsilon))} (-1)^{\nu\epsilon} , & \text{if } j = n_\nu . \end{cases}$$

The Weingarten function $\text{Wg}_{\text{SP}(2N)}(\mathbf{m}, \mathbf{n})$ is the (\mathbf{m}, \mathbf{n}) -entry of the inverse of the Gram matrix of the extension of a (as above) with respect to the basis $\{\theta_{\mathbf{m}} \mid \mathbf{m} \in \mathcal{PP}(2l)\}$. On $\mathcal{PP}(2) \times \mathcal{PP}(2)$ one obtains $\text{Wg}_{\text{SP}(2N)} = \frac{1}{2N}$, and on $\mathcal{PP}(4) \times \mathcal{PP}(4)$ it is given by (27) above.

Proposition 2 *Let $\phi, \psi \in \mathcal{F}(k, N)$, $\alpha, \beta \in \mathcal{F}(k, \{0, 1\})$. Then*

$$\int_{\text{SP}(2N)} dU \prod_{j=1}^k U_{\phi(j)+N\alpha(j), \psi(j)+N\beta(j)} = \sum_{\mathbf{m}, \mathbf{n} \in \mathcal{PP}(k)} \text{Wg}_{\text{SP}(2N)}(\mathbf{m}, \mathbf{n}) (-1)^{\sum_{\nu=1}^l \alpha(m_\nu(\mathbf{m})) + \beta(m_\nu(\mathbf{n}))} \\ 1_{\mathcal{F}(\mathbf{m}, 2N)}(\phi) 1_{\mathcal{A}(\mathbf{m}, 2N)}(\alpha) 1_{\mathcal{F}(\mathbf{n}, 2N)}(\psi) 1_{\mathcal{A}(\mathbf{n}, 2N)}(\beta) .$$

Finally let us turn to the unitary case. Let S_k denote the full symmetric permutation group of the set $\{1, \dots, k\}$, and e_1^*, \dots, e_N^* the dual to the basis e_1, \dots, e_N on $V = \mathbb{C}^N$ w.r.t. the standard scalar product c . If $N \geq k$, then the space of $U(N)$ -invariants of $V^{\otimes k} \otimes (V^*)^{\otimes k}$ (with the contragredient representation on the dual space) admits a basis $\{C_\pi \mid \pi \in S_k\}$ where

$$C_\pi = \sum_{\phi \in \mathcal{F}(k, N)} e_{\phi(\pi^{-1}(1))} \otimes \dots \otimes e_{\phi(\pi^{-1}(k))} \otimes e_{\phi(1)}^* \otimes \dots \otimes e_{\phi(k)}^* .$$

The Weingarten function $\text{Wg}_{U(N)}(\sigma, \tau)$ is the (σ, τ) -entry of the inverse of the Gram matrix of the extension of the scalar product c w.r.t. the basis $\{C_\pi \mid \pi \in S_k\}$. Then on $S_1 \times S_1$ one has $\text{Wg}_{U(N)} = \frac{1}{N}$, while on $S_2 \times S_2$ the unitary Weingarten function is given by

$$\frac{1}{N(N^2 - 1)} \begin{pmatrix} N & -1 \\ -1 & N \end{pmatrix} .$$

Proposition 3 *Let $\phi, \psi, \phi', \psi' \in \mathcal{F}(k, N)$. Then*

$$\int_{U(N)} dU \prod_{j=1}^k U_{\phi(j), \psi(j)} \overline{U}_{\phi'(j), \psi'(j)} = \sum_{\sigma, \tau \in S_k} \text{Wg}_{U(N)}(\sigma, \tau) \delta(\phi, \sigma\phi') \delta(\psi, \tau\psi') .$$

Let us conclude with a comment on higher order Weingarten functions. Their definitions are given here have the merit that they can be easily motivated in an invariant theoretic context. Their calculation, in particular, for higher orders, can be simplified by remarking, e.g., that $\text{Wg}_{U(N)}(\sigma, \tau)$ depends only on the cycle type of $\sigma\tau^{-1}$. This leads to sophisticated and computationally feasible expansions of Wg at least in the unitary and orthogonal cases [CM].

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